


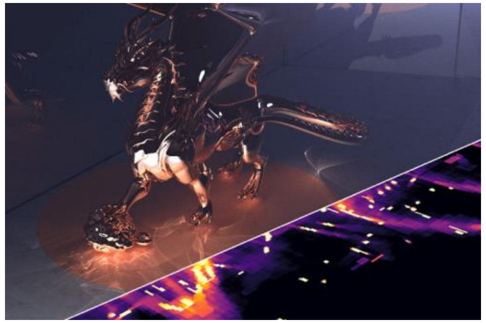

Homepage

Light Transport with Specular Constraints

Modeling, Solving, and Bounding

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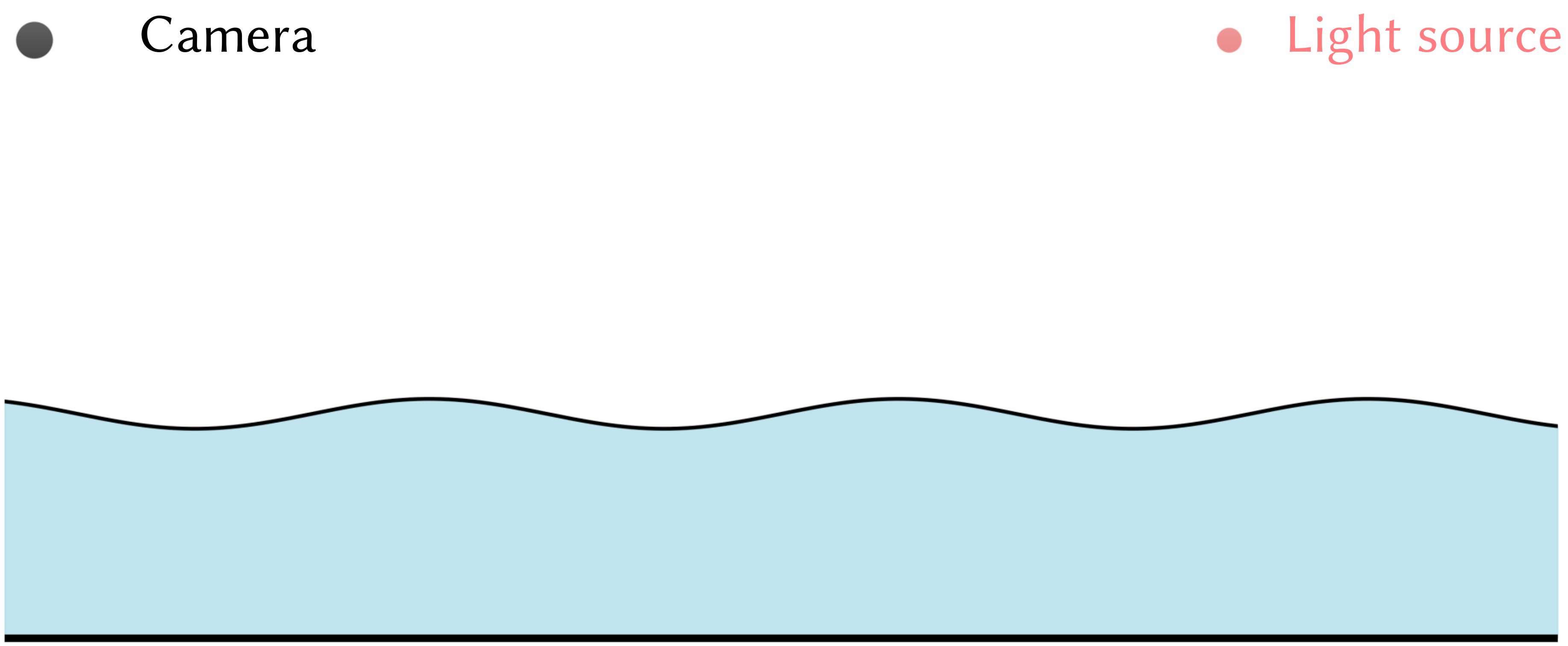
Specular light transport is important

- Caustics, a typical effect caused by specular paths

Path tracing

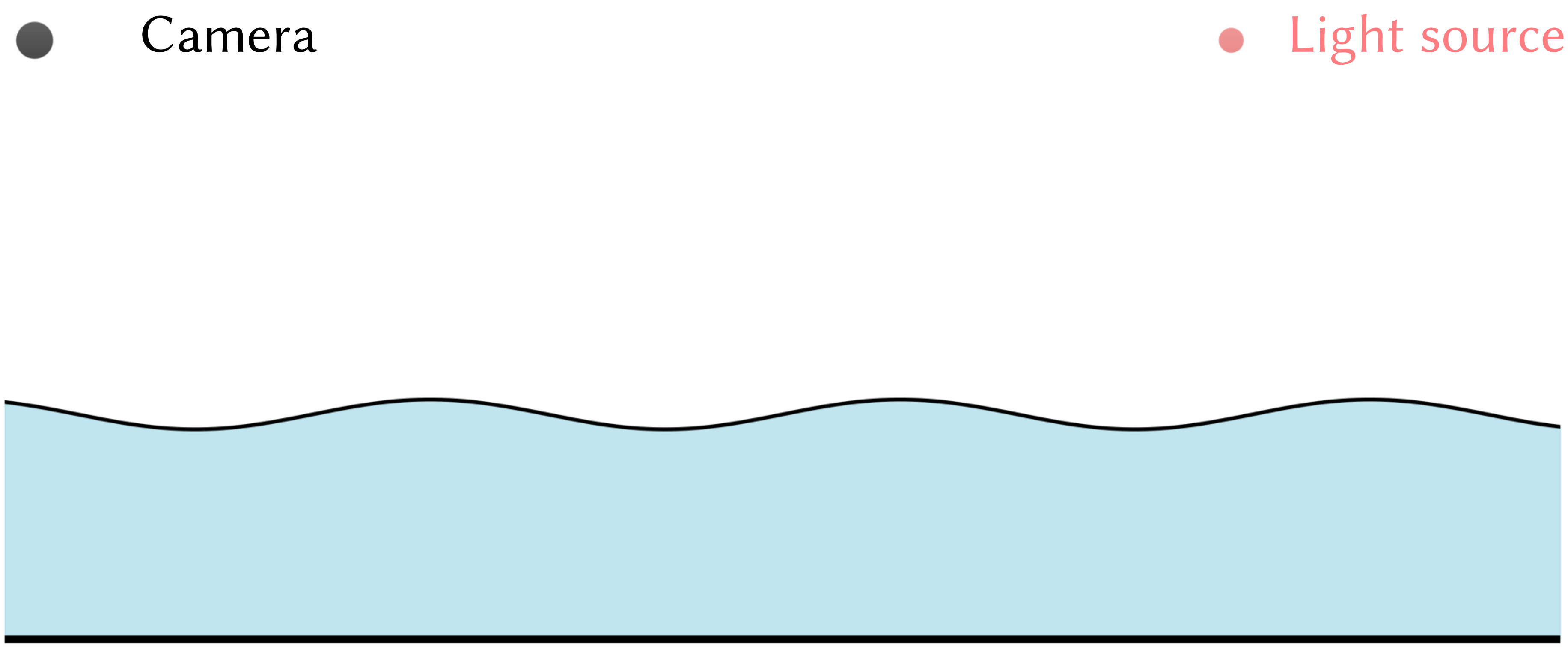
[Kajiya 1986]



Hard to connect to the light source

Path tracing

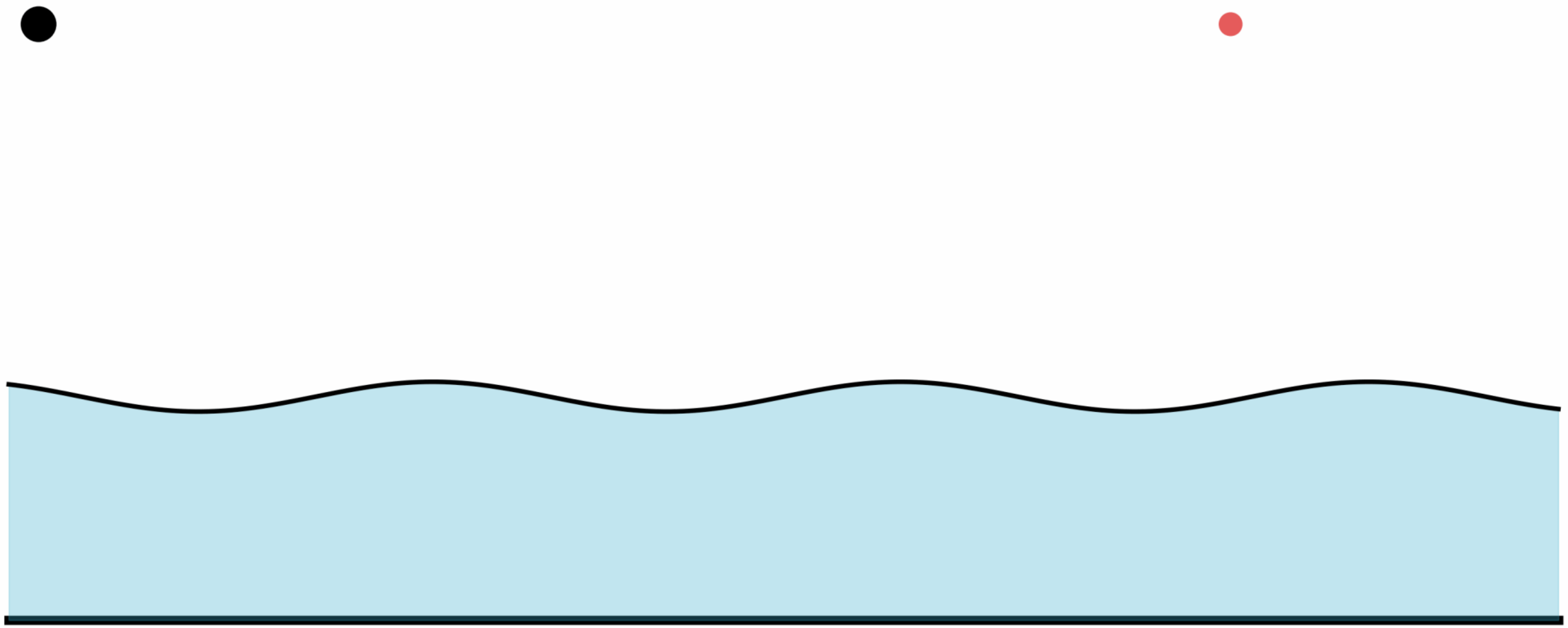
[Kajiya 1986]



Incident radiance: near-delta distributions

Photon mapping

[Jensen 1996]



Bias due to spatial relaxation

Rendering sharp caustics is difficult!

Path guiding [Müller et al. 2017]

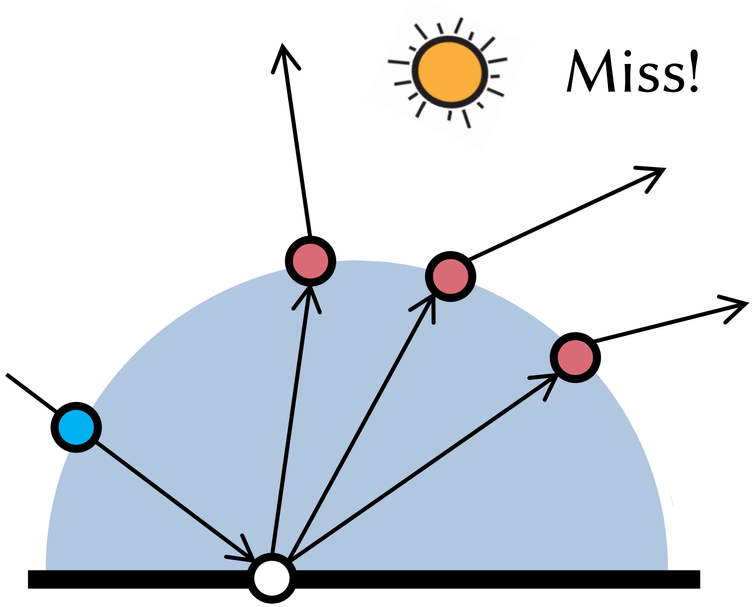
- Extremely slow to converge

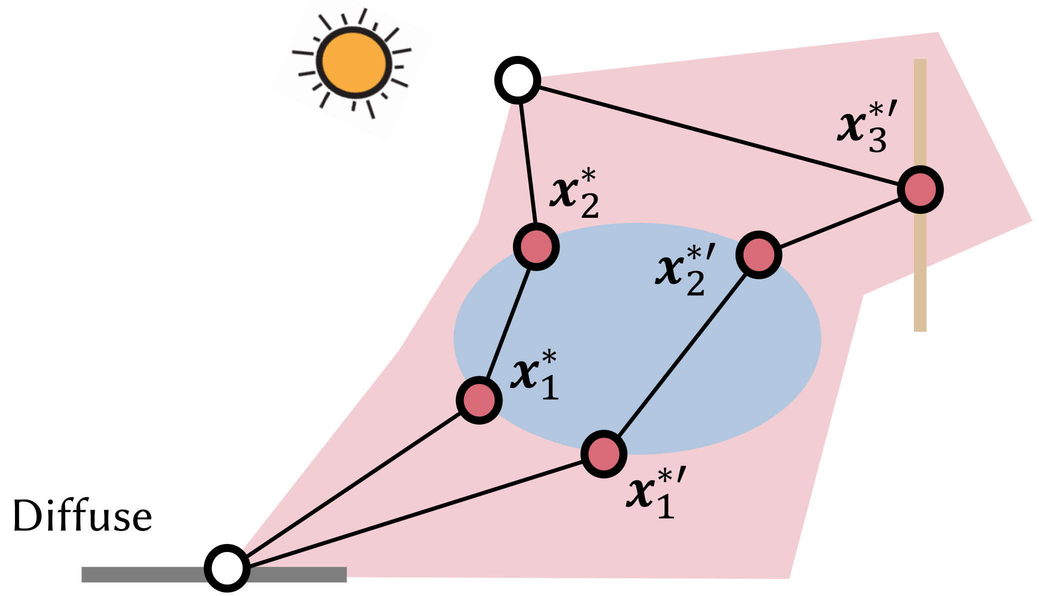
SPPM [Hachisuka et al. 2009]

- Blurring and noise

Specular path

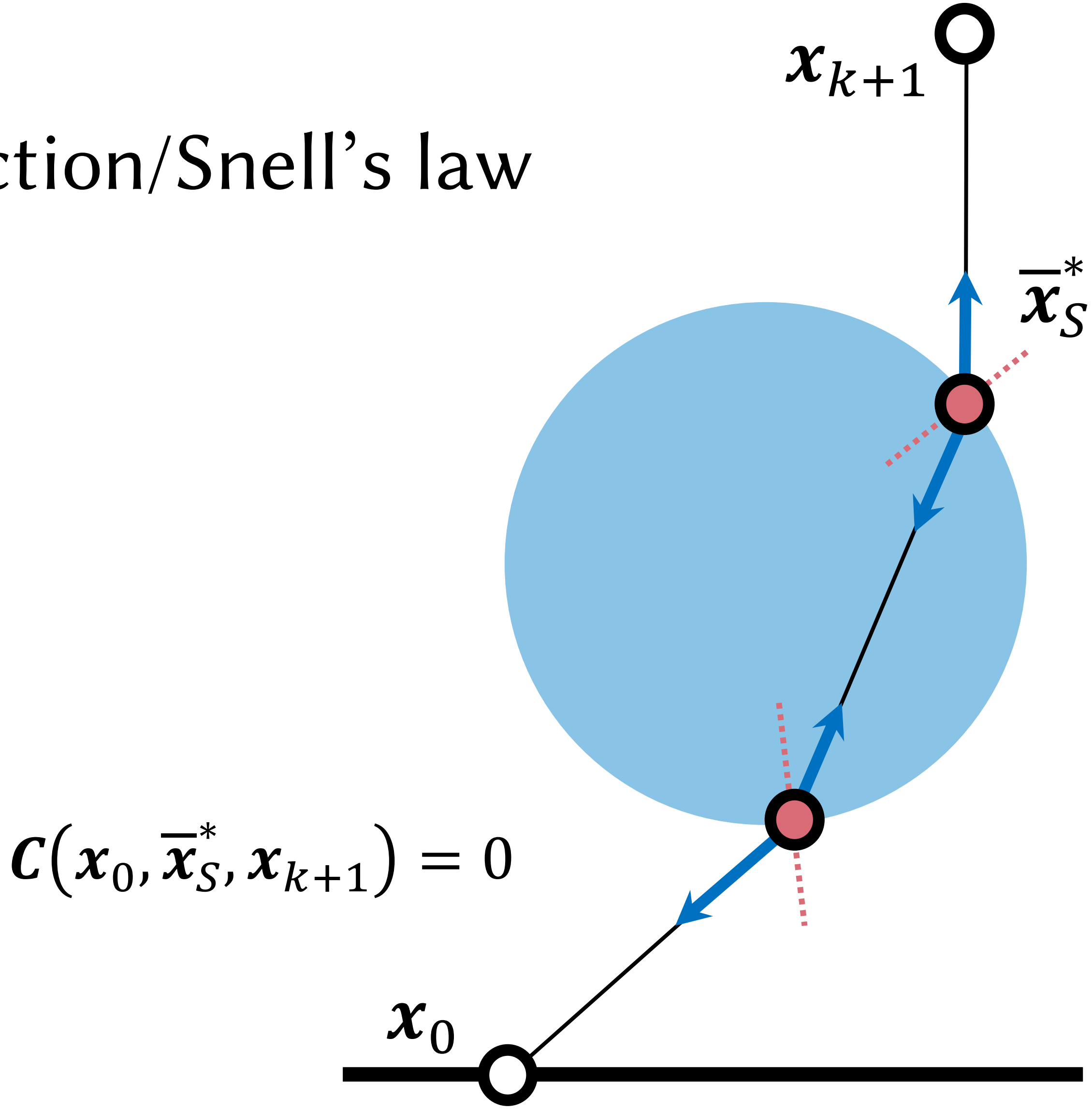
- Local sampling fails to reach a (near-)**point light source**
- Specialized methods connect **endpoints** with **specular vertices**
- **Problem: How to connect?**



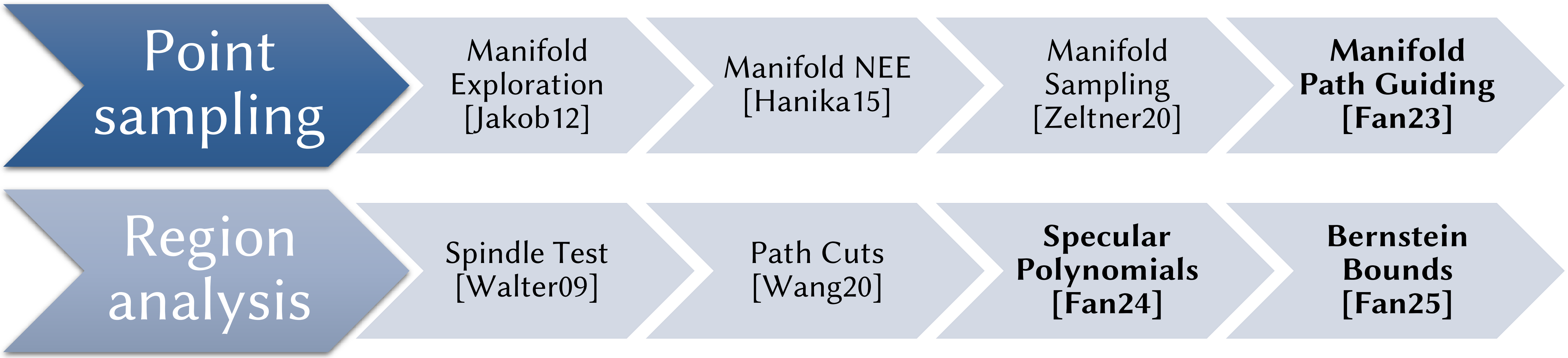


Specular constraints

- Specular vertices satisfy the reflection/Snell's law
- Just solve the equations!
  - k specular vertices
  - $2k$ variables
  - $2k$ equations
- Difficult to solve!



Recent advancements

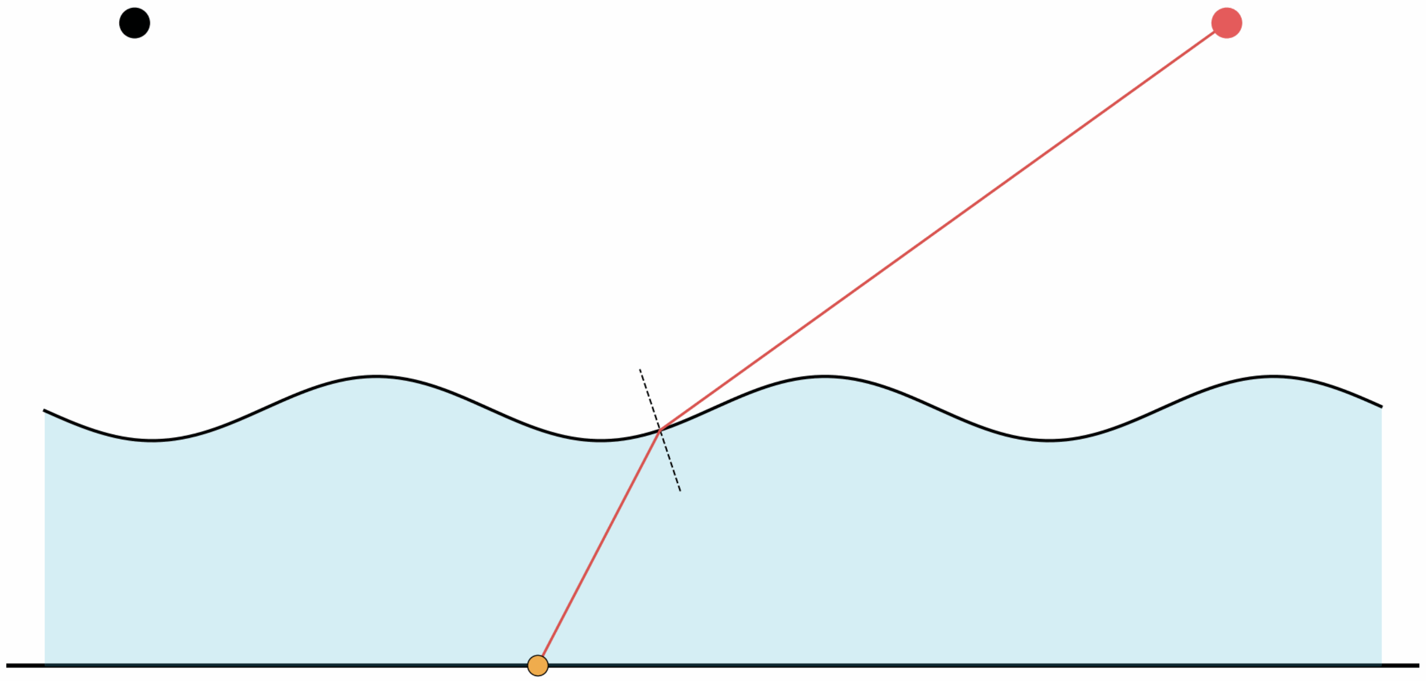


Other works

Analytic GGX integral [Loubet20], Path-cut-based guiding [Li22]
Large jump for manifold sampling [Jhang22], Specular path reuse [Xu23], Neural manifold sampling [Yu23]
Photon-guided sampling [Lee24], Single-bounce dimension reduction [Granizo-Hidalgo24], Position-normal manifold clustering [Wu25]

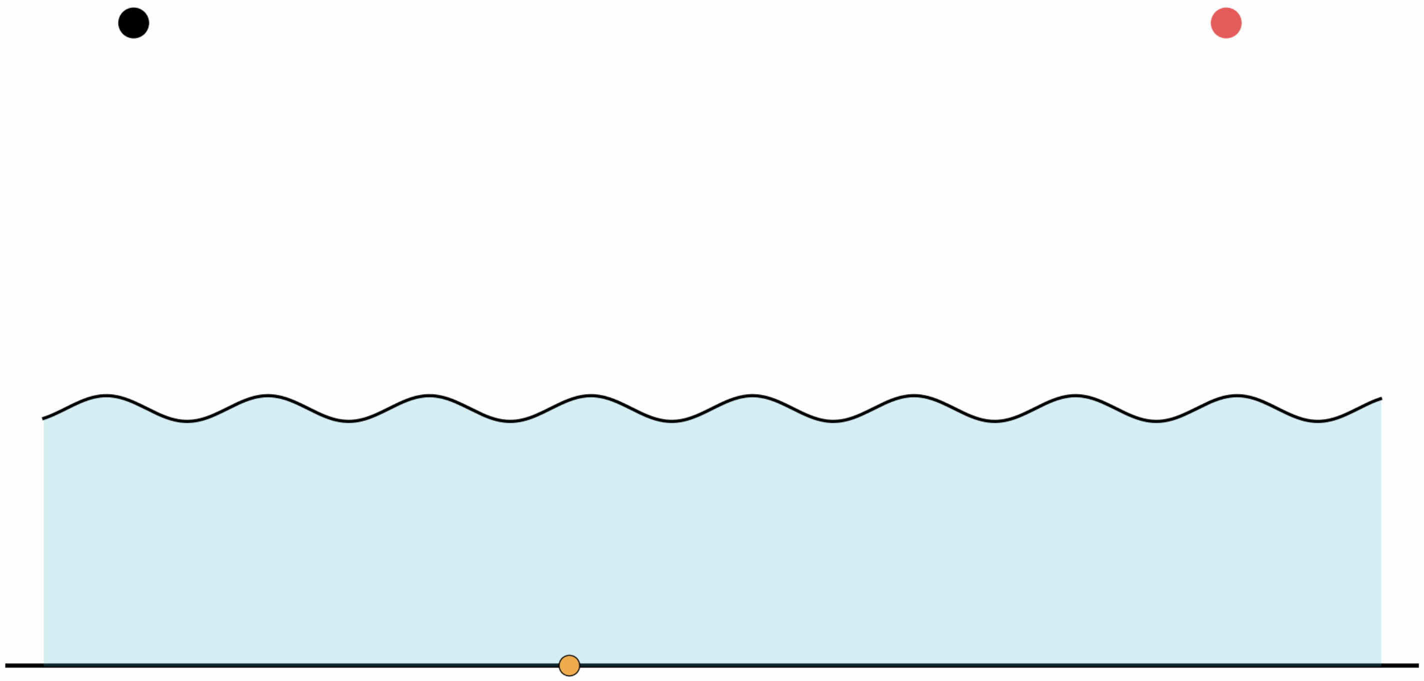
Manifold: deterministic init.

- Manifold next event estimation [Hanika et al. 2015]



Manifold: uniform stochastic sampling

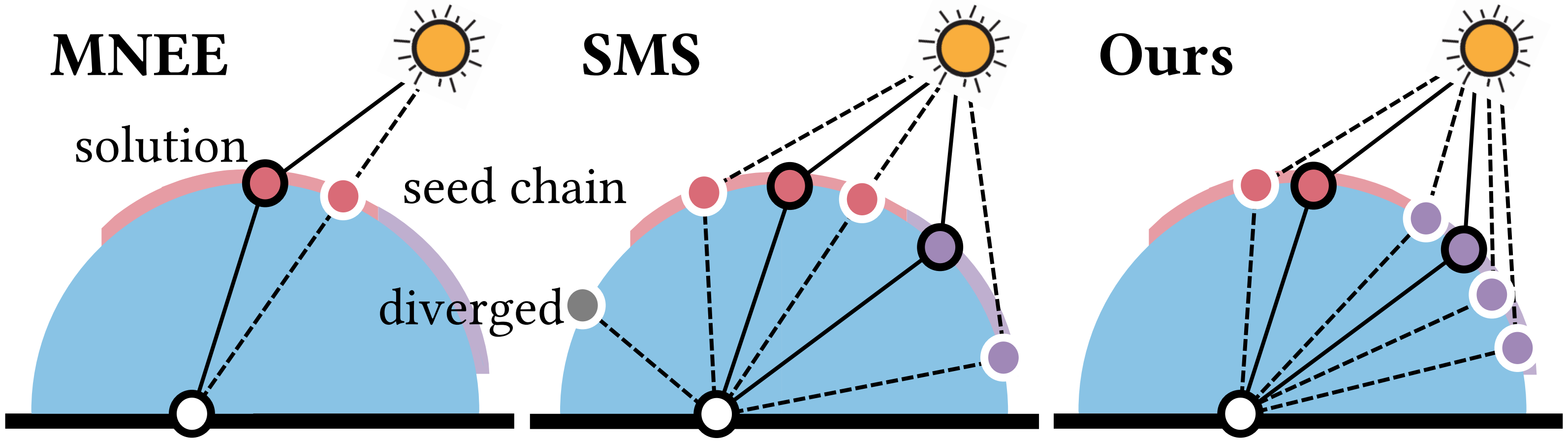
- Specular manifold sampling [Zeltner et al. 2020]



Manifold path guiding: motivation

- MNEE finds **at most one solution**, resulting in **energy loss**
- SMS **uniformly** samples the seeds, which leads to **high variance**

- **Goal**: find paths that are not only **admissible** but also **"important"**



Fitting

Data-driven Modeling


Manifold Path Guiding for Importance Sampling Specular Chains

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1 INTRODUCTION

Monte Carlo (MC) integration using stochastic samples has long been the de facto solution to the problem of physically-based light transport simulation [Christensen and Jarosz 2016; Fascione et al. 2018; Keller et al. 2015]. Over the past decades, great efforts have

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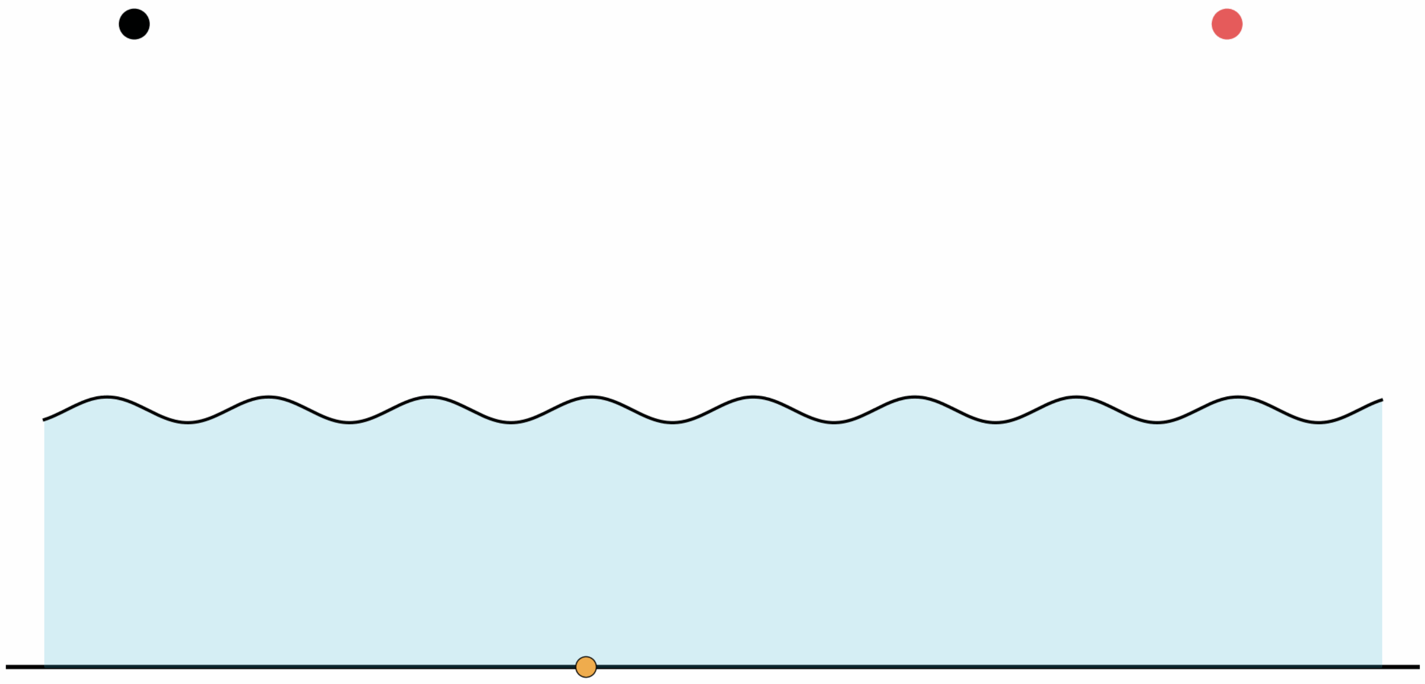
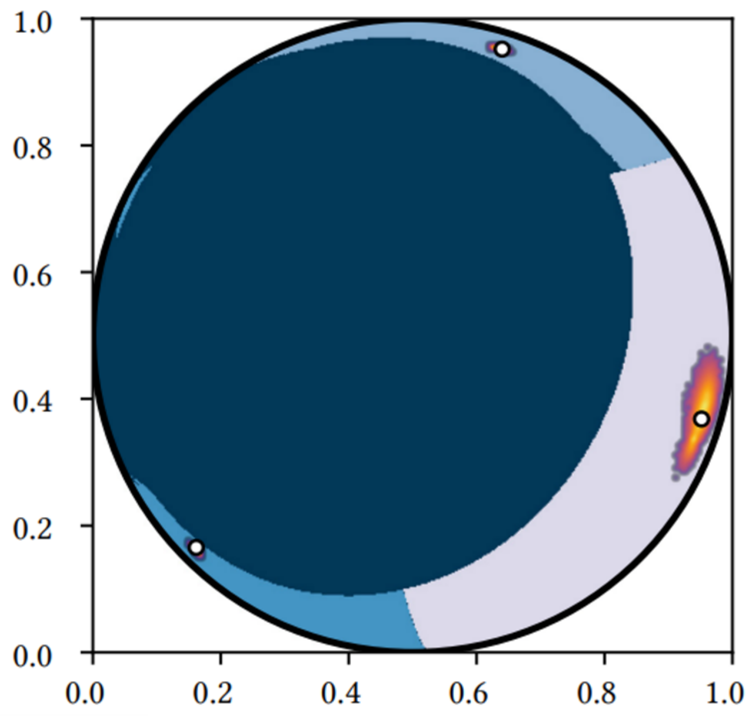
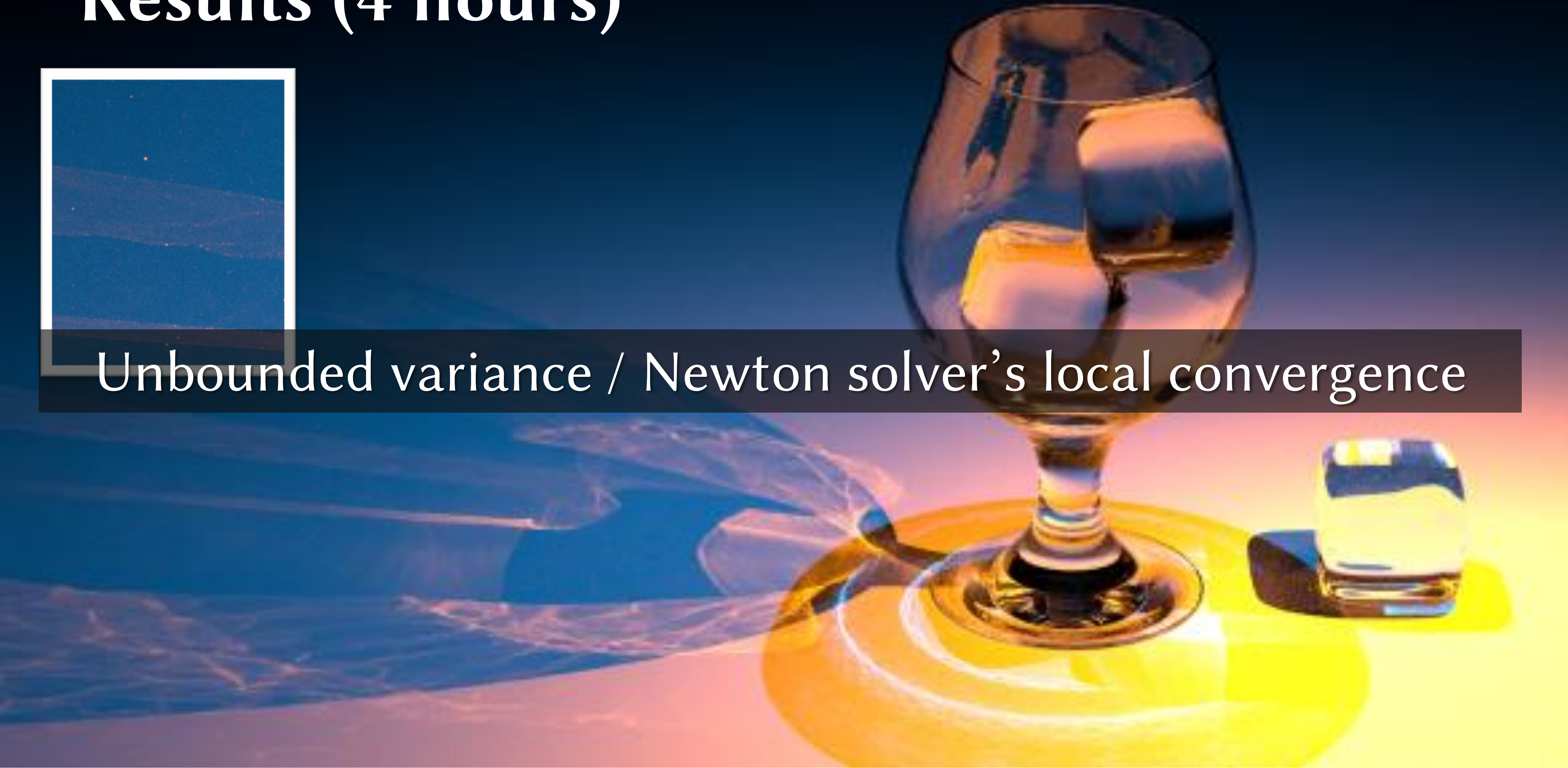
Manifold: importance sampling

- Manifold path guiding [Fan et al. 2023]





Results (4 hours)



Unbounded variance / Newton solver’s local convergence

Solving

Analytic Modeling


Specular Polynomials

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SPPM MPG Ours

Fig. 1. We render a shop window scene featuring challenging caustics and complex visibility, using our pipeline based on specular polynomials. The caustics stem from colored point light sources placed inside a dielectric object, and the whole scene is viewed through a transparent window. Such a configuration makes most existing rendering algorithms fail, while our method succeeds in reproducing the stunning light transport effect. The insets show equal-time (10 min) comparisons against Stochastic Progressive Photon Mapping (SPPM) [Hachisuka and Jensen 2009] and Manifold Path Guiding (MPG) [Fan et al. 2023].

Finding valid light paths that involve specular vertices in Monte Carlo rendering requires solving many non-linear, transcendental equations in high-dimensional space. Existing approaches heavily rely on Newton iterations in path space, which are limited to obtaining at most a single solution each time and easily diverge when initialized with improper seeds.

We propose *specular polynomials*, a Newton iteration-free methodology for finding a complete set of admissible specular paths connecting two arbitrary endpoints in a scene. The core is a reformulation of specular constraints into polynomial systems, which makes it possible to reduce the task to a univariate root-finding problem. We first derive bivariate systems utilizing rational coordinate mapping between the coordinates of consecutive vertices. Subsequently, we adopt the hidden variable resultant method for variable elimination, converting the problem into finding zeros of the determinant of univariate matrix polynomials. This can be effectively solved through Laplacian expansion for one bounce and a bisection solver for more bounces.

Our solution is generic, completely deterministic, accurate for the case of one bounce, and GPU-friendly. We develop efficient CPU and GPU implementations and apply them to challenging glints and caustic rendering. Experiments on various scenarios demonstrate the superiority of specular polynomial-based solutions compared to Newton iteration-based counterparts. Our implementation is available at https://github.com/mollnn/spoly.

CCS Concepts: • **Computing methodologies → Ray tracing.**

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Why Polynomials?

- Root-solving with **global** convergence
  - **Root isolation** by computing derivatives recursively
  - **Eigenvalue**: QR/QZ decomposition

- **Key idea**: derive (1D) polynomial formulations

Specular Polynomials

- Multivariate polynomial constraints $F(\boldsymbol{u}_{i-1}, \boldsymbol{u}_i, \boldsymbol{u}_{i+1}) = 0$

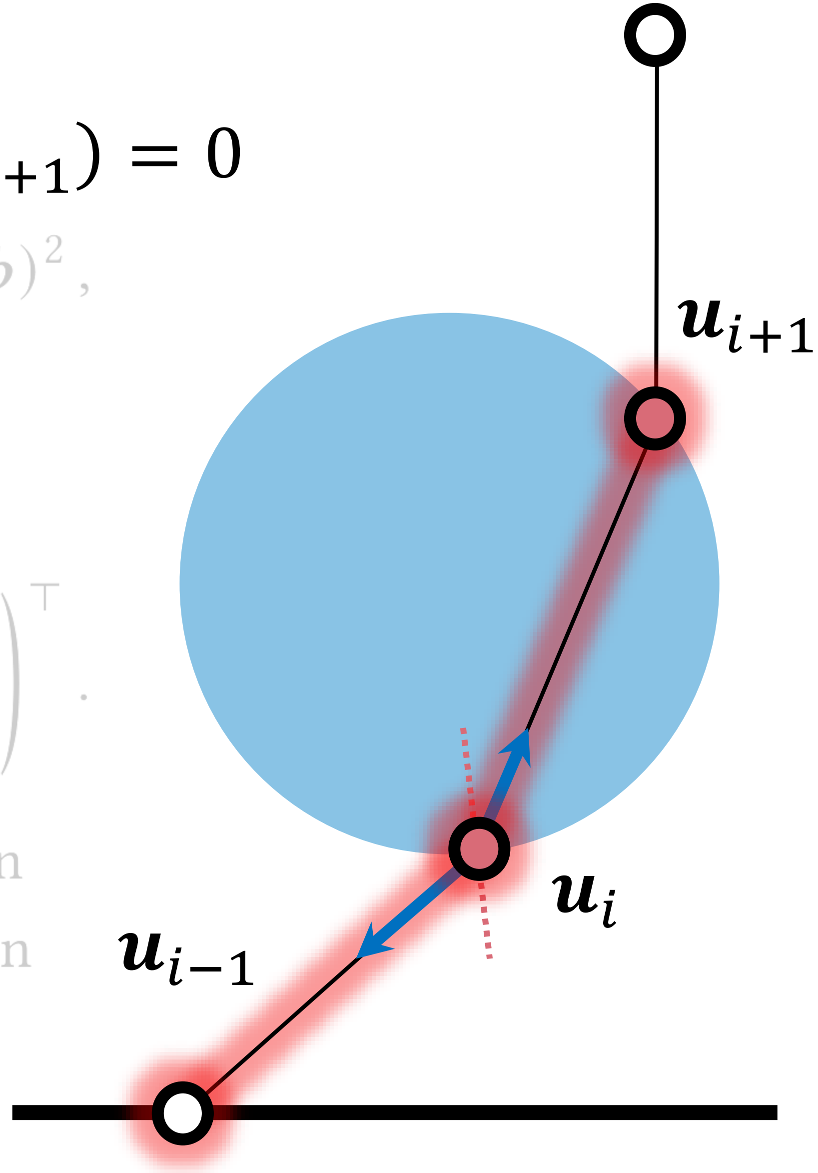
$$\begin{cases} F = \boldsymbol{d}_{k-1}^2 \left((\boldsymbol{d}_{k-2} \times \boldsymbol{n}_{k-1}) \cdot \boldsymbol{b} \right)^2 - \eta^2 \boldsymbol{d}_{k-2}^2 \left((\boldsymbol{d}_{k-1} \times \boldsymbol{n}_{k-1}) \cdot \boldsymbol{b} \right)^2, \\ G = (\boldsymbol{d}_{k-2} \times \boldsymbol{n}_{k-1}) \cdot \boldsymbol{d}_{k-1}. \end{cases}$$

- Rational coordinate mapping $\boldsymbol{u}_{i+1} = f(\boldsymbol{u}_i, \boldsymbol{u}_{i-1})$

$$\boldsymbol{u}_{i+1} = \left(\frac{(\tilde{\boldsymbol{d}}_i \times \boldsymbol{e}_{i+1,2}) \cdot (\boldsymbol{x}_i - \boldsymbol{p}_{i+1,0})}{(\tilde{\boldsymbol{d}}_i \times \boldsymbol{e}_{i+1,2}) \cdot \boldsymbol{e}_{i+1,1}}, \frac{((\boldsymbol{x}_i - \boldsymbol{p}_{i+1,0}) \times \boldsymbol{e}_{i+1,1}) \cdot \tilde{\boldsymbol{d}}_i}{(\tilde{\boldsymbol{d}}_i \times \boldsymbol{e}_{i+1,2}) \cdot \boldsymbol{e}_{i+1,1}} \right)^\top .$$

$$\tilde{\boldsymbol{d}}_i = \begin{cases} (\boldsymbol{n}_i \cdot \boldsymbol{n}_i)\boldsymbol{d}_{i-1} - 2(\boldsymbol{d}_{i-1} \cdot \boldsymbol{n}_i)\boldsymbol{n}_i, & \text{for reflection} \\ \eta_i' \left((\boldsymbol{n}_i \cdot \boldsymbol{n}_i)\boldsymbol{d}_{i-1} - (\boldsymbol{d}_{i-1} \cdot \boldsymbol{n}_i)\boldsymbol{n}_i \right) - \sqrt{\beta_i}\boldsymbol{n}_i, & \text{for refraction} \end{cases}$$

$$\beta_i = (1 - \eta_i'^2)(\boldsymbol{n}_i \cdot \boldsymbol{n}_i)(\boldsymbol{d}_{i-1} \cdot \boldsymbol{d}_{i-1}) + \eta_i'^2 (\boldsymbol{d}_{i-1} \cdot \boldsymbol{n}_i)^2 .$$



Resultant elimination for two polynomials

- The necessary condition of
  - the existence of common roots of
  - **two** polynomials
- Bézout's resultant

$$r(v_1) = \det \boldsymbol{R}(v_1)$$

$$\begin{cases} a(u_1, v_1) = \sum_{i=0}^{n} a_i(v_1)u_1^i = 0, \\ b(u_1, v_1) = \sum_{i=0}^{n} b_i(v_1)u_1^i = 0. \end{cases}$$

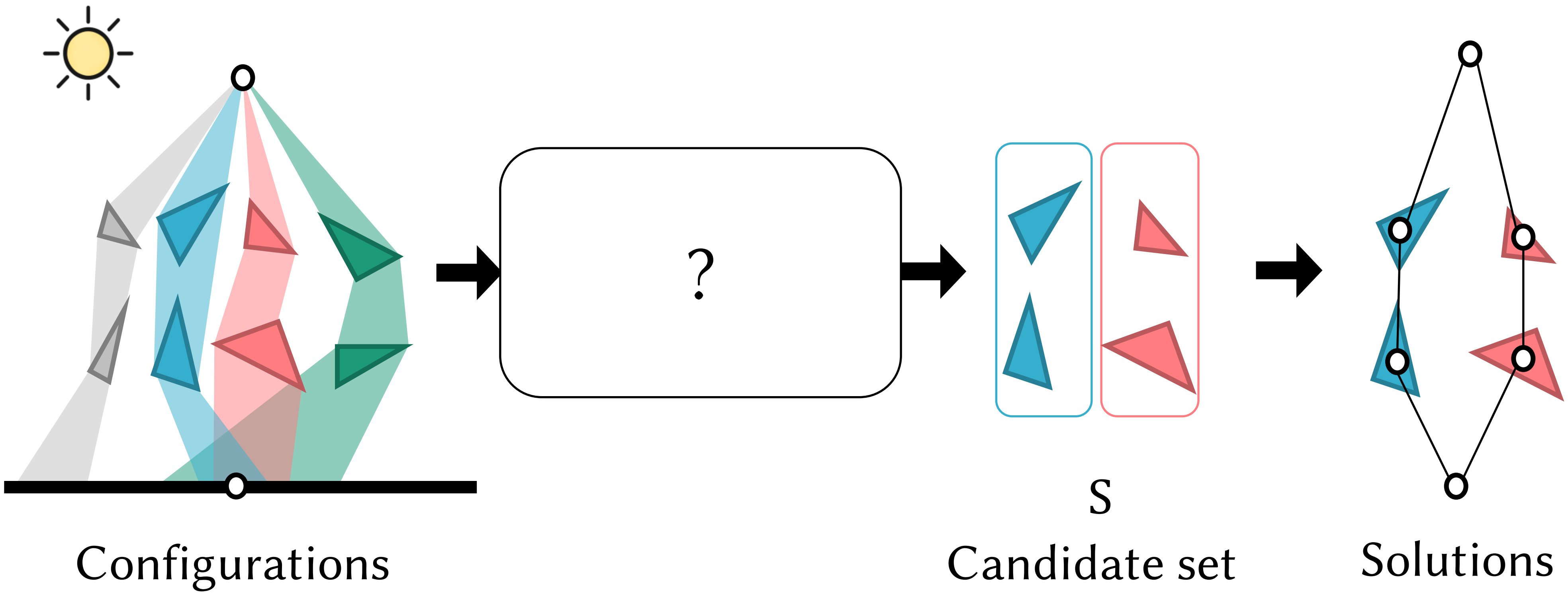
$$R_{i,j} = \sum_{k=0}^{\min(i,n-1-j)} \left(a_{i-k}(v_1)b_{j+1+k}(v_1) - b_{i-k}(v_1)a_{j+1+k}(v_1)\right)$$

Only in v_1 now!

Deterministic search for specular paths

- **Select triangle tuples** first (focus in **Fan et al. 2025**, also using SP)
- Then **find solutions** using Newton's method or polynomial solvers

[Walter et al. 2009] [Wang et al. 2020] **[Fan et al. 2024]**

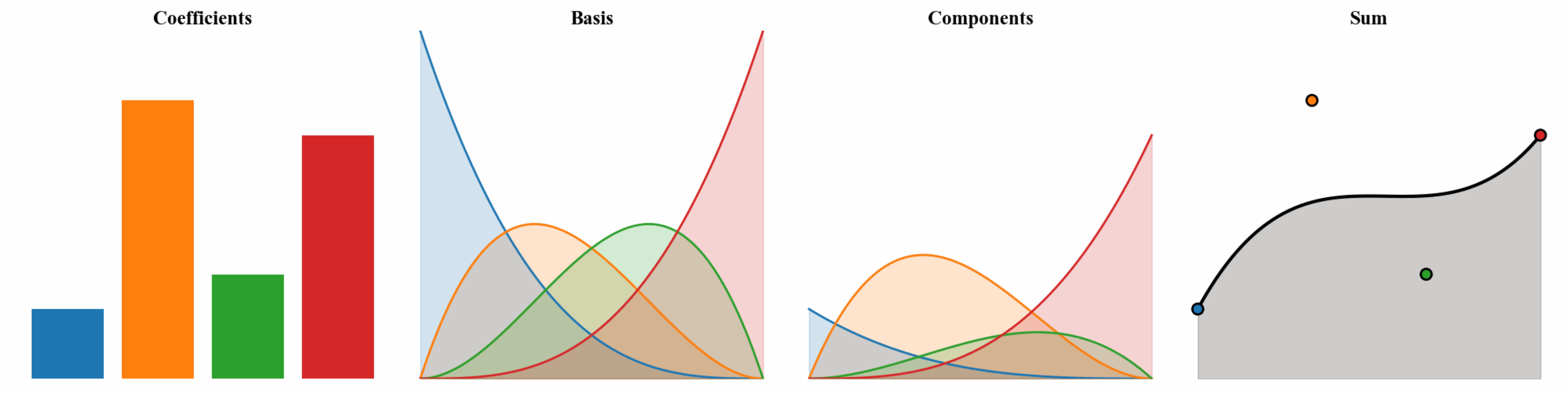


Bounding

Analytic Modeling

Bernstein polynomial basis

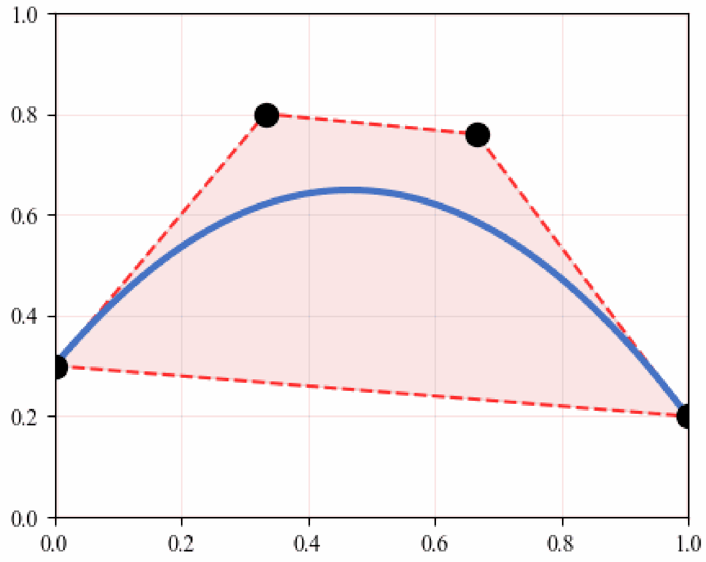
$$f(x) = \sum_{i=0}^{n} b_i^f x^i (1-x)^{n-i}$$

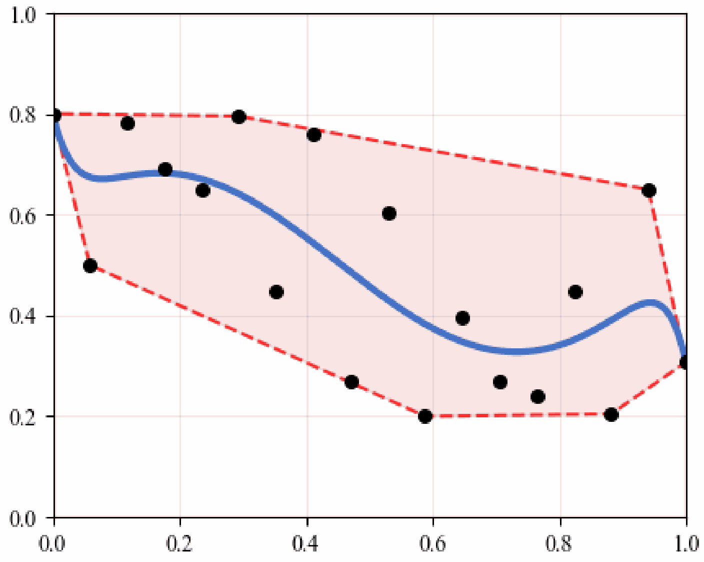


Bounding the range of polynomials

$$f(x) = \sum_{i=0}^{n} b_i^f x^i (1-x)^{n-i}$$

- $(x, f(x))$ falls in the convex hull formed by $\left\{ \left(\frac{i}{n-1}, b_i^f \right) \middle| 0 \le i \le n \right\}$

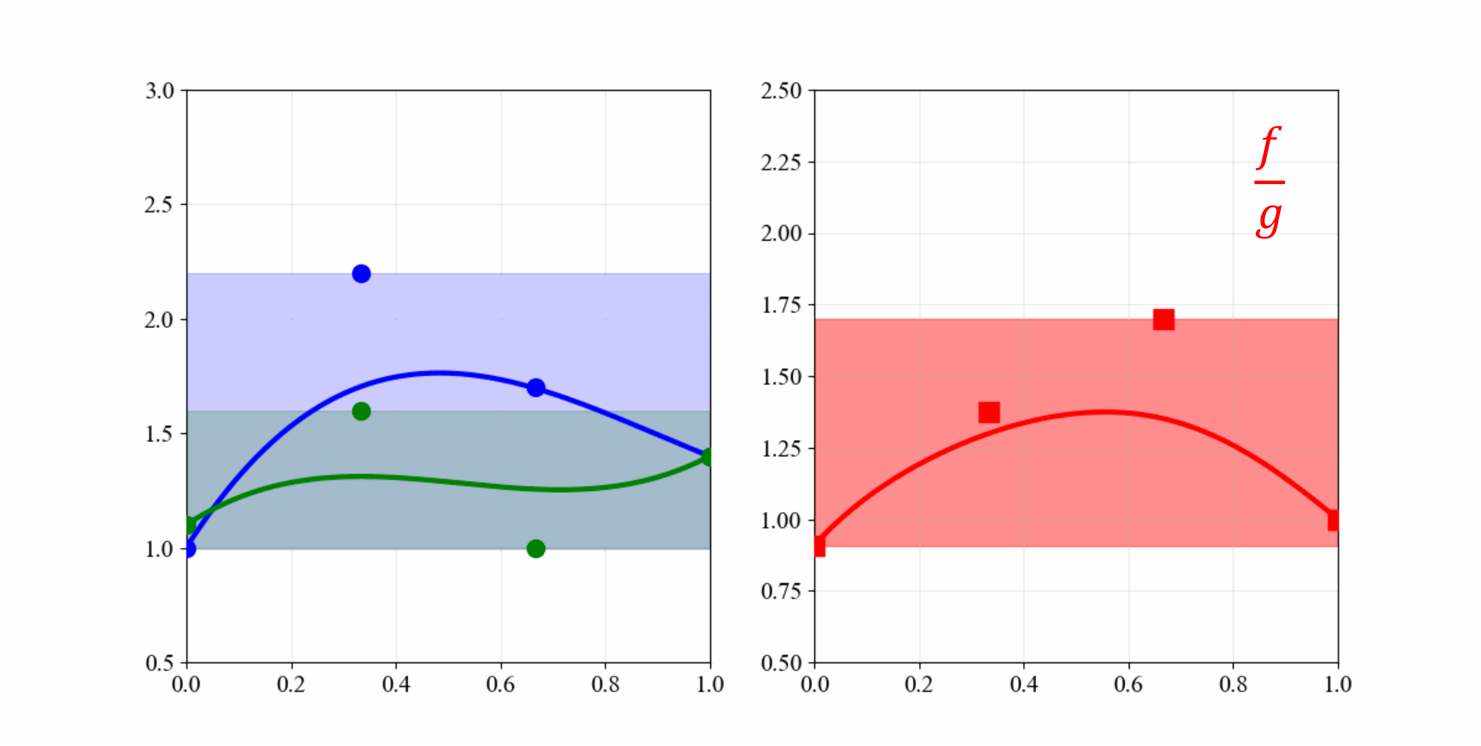




Bounding the range of rational functions

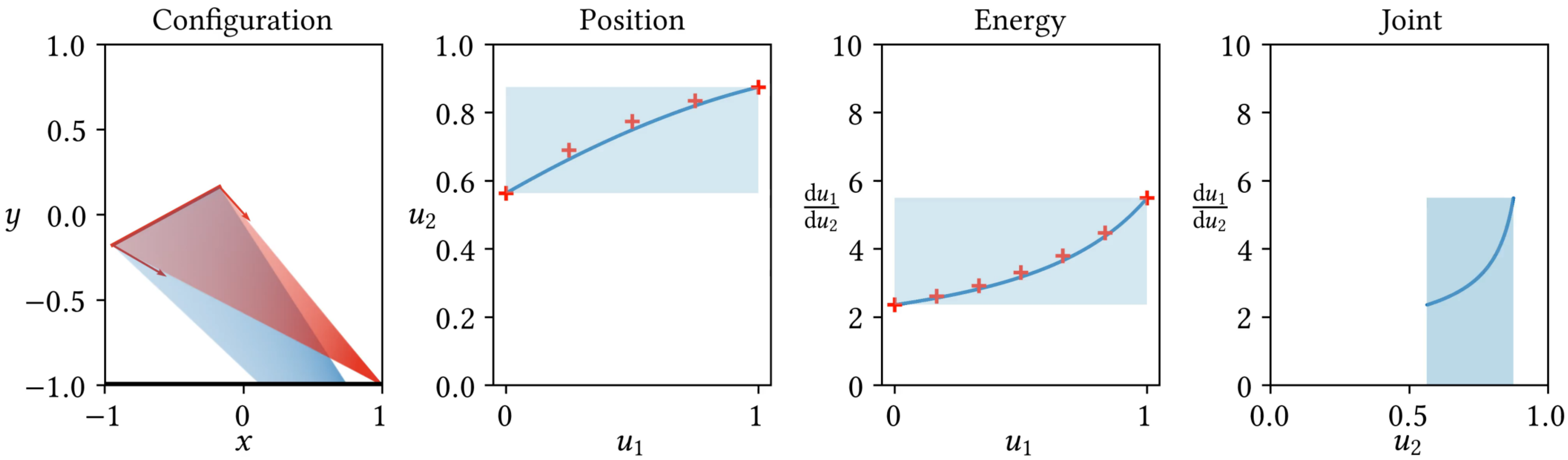
$$\min_{i=0}^{n} \frac{b_i^f}{b_i^g} \leq \frac{f(x)}{g(x)} \leq \max_{i=0}^{n} \frac{b_i^f}{b_i^g} \quad (g(x) \neq 0)$$

[Narkawicz et al. 2012]



Bernstein bounds for caustics

- Represent caustics with Bernstein polynomials
- Obtain their bounds using the bounding property



Summary

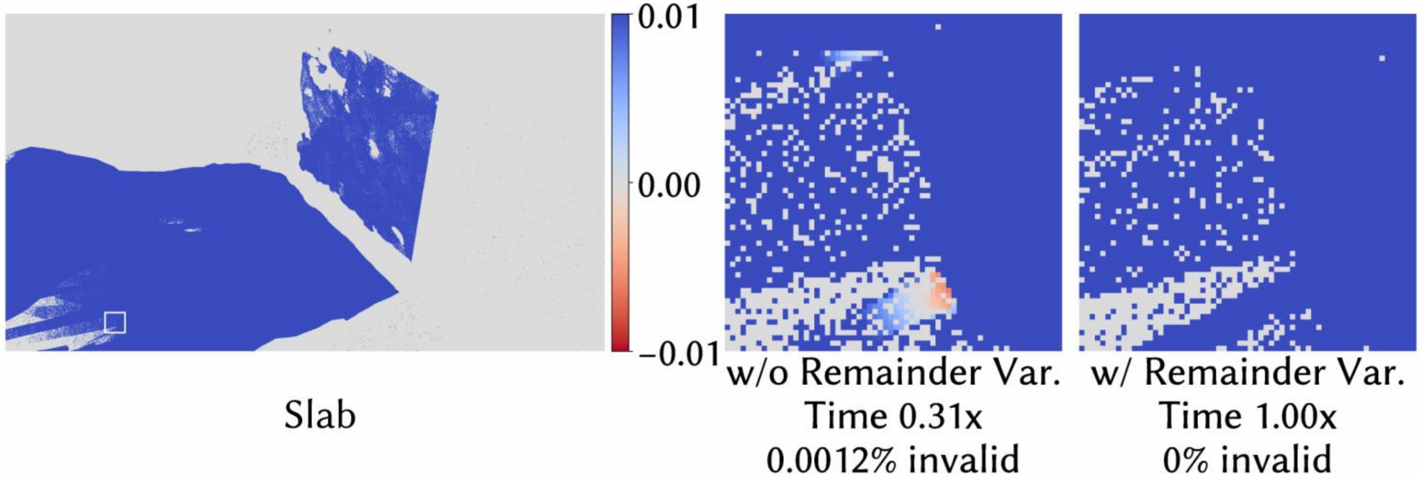
- **Fitting distributions for the constrain solutions (23')**
  - General
  - *Needs training/initial distributions, suffers from outlier cases, so…*

- **Reformulating constraints into polynomials (24')**
  - Thus enabling many mathematical tools
  - **Solving (24')**
    - Resultant elimination | Root isolation/QZ decomposition
    - *Too slow due to enumeration, so…*

  - **Bounding (25')**
    - Bernstein polynomials | Regression with remainders
    - *Producing an approximated distribution of the constrain solutions*

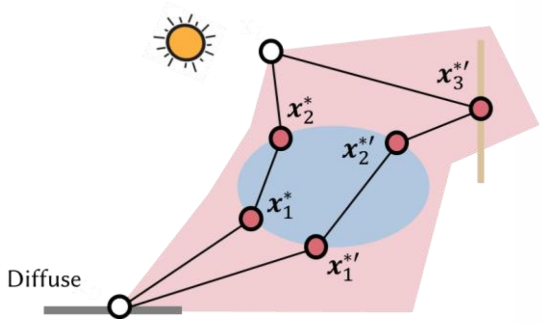
Open questions

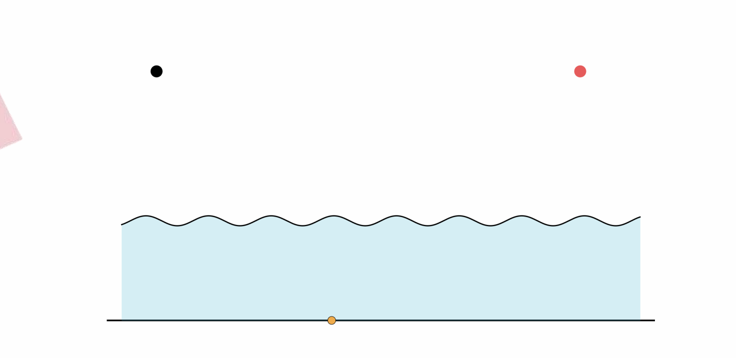
Performance

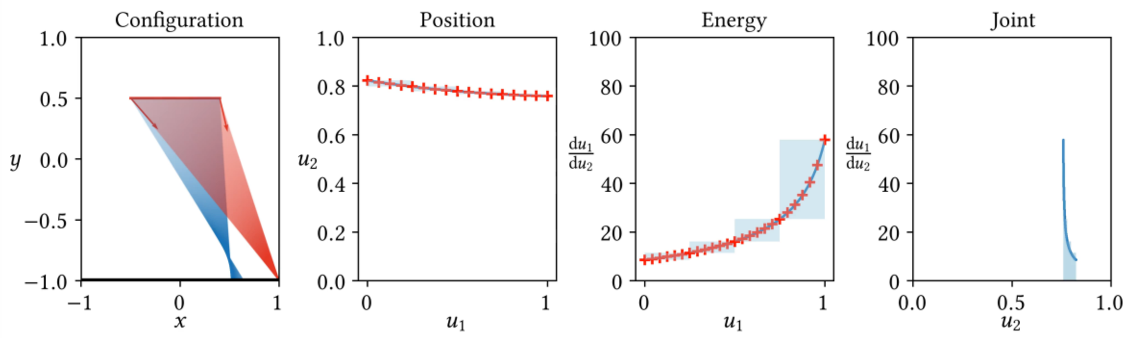


- **We focus on**
  - a theoretically sound solution
  - and accurate modeling

- **Allow violations/approximations** to be more efficient
  - How to strike a balance?







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Homepage

Thank You!

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Contact me if you have any questions!