

Homepage

Light Transport with Specular Constraints

Modeling, Solving, and Bounding

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Specular light transport is important

- Caustics, a typical effect caused by specular paths



Path tracing

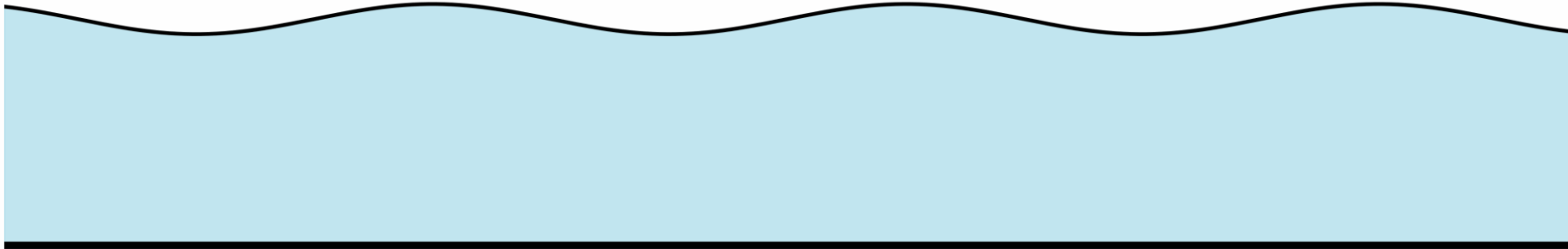
[Kajiya 1986]



Camera



Light source



Hard to connect to the light source

Path tracing

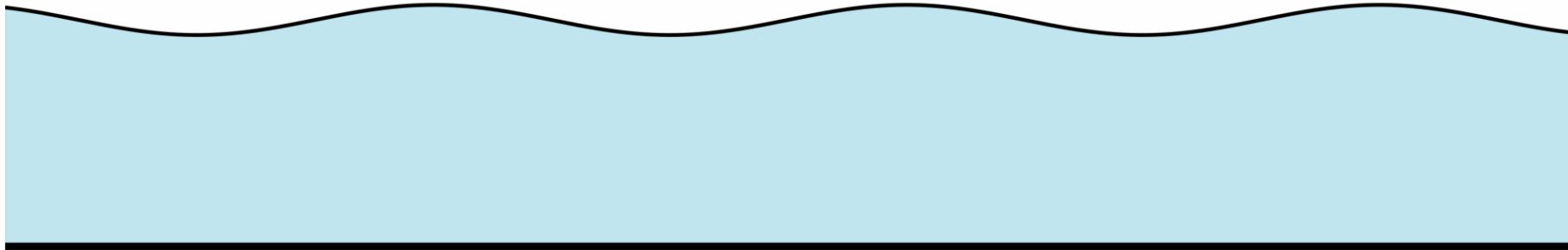
[Kajiya 1986]



Camera



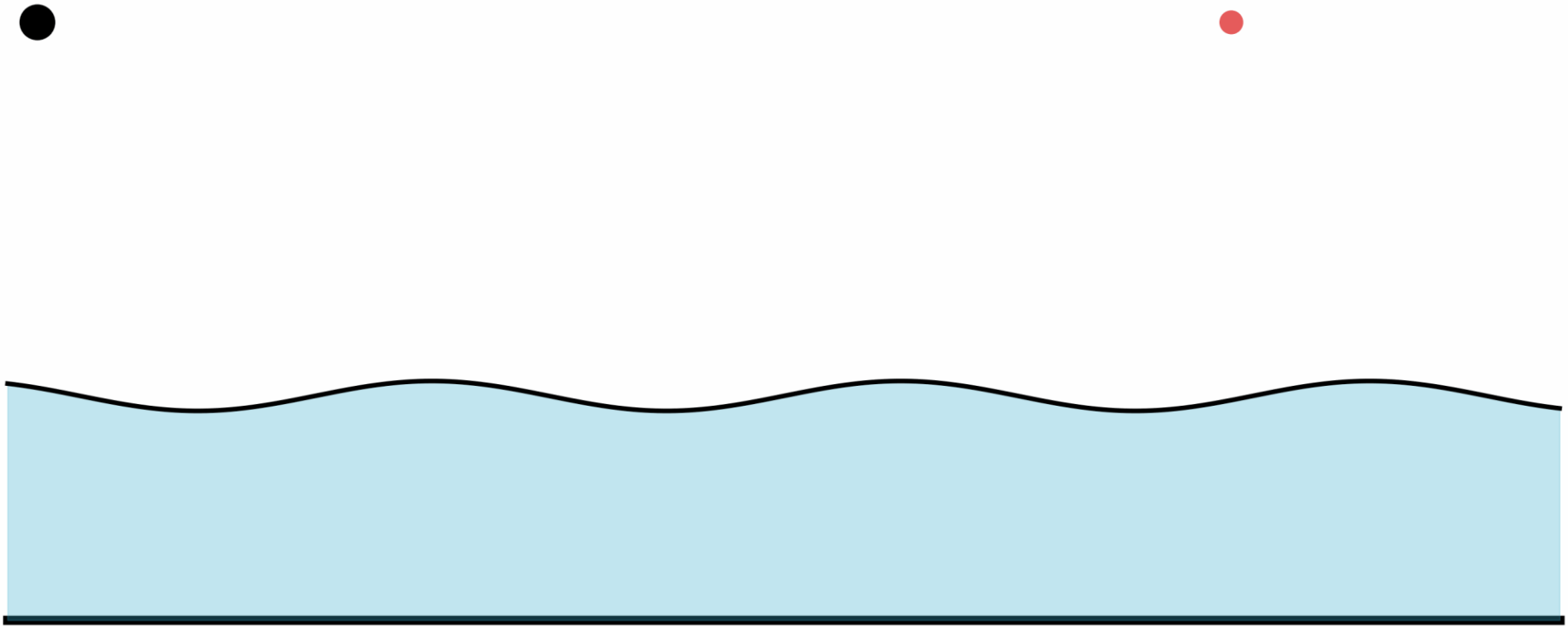
Light source



Incident radiance: near-delta distributions

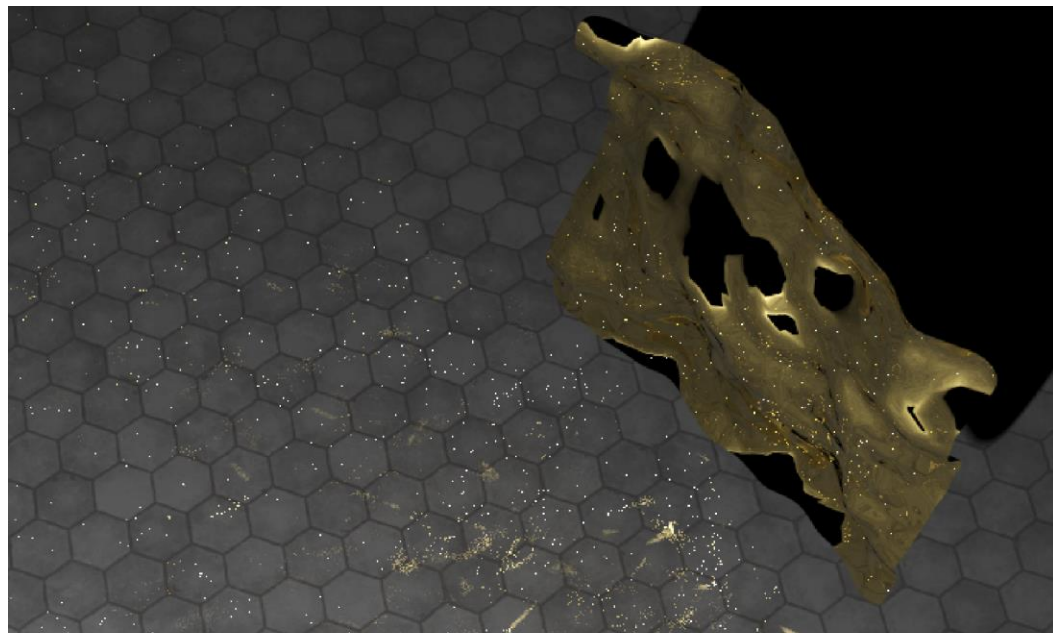
Photon mapping

[Jensen 1996]



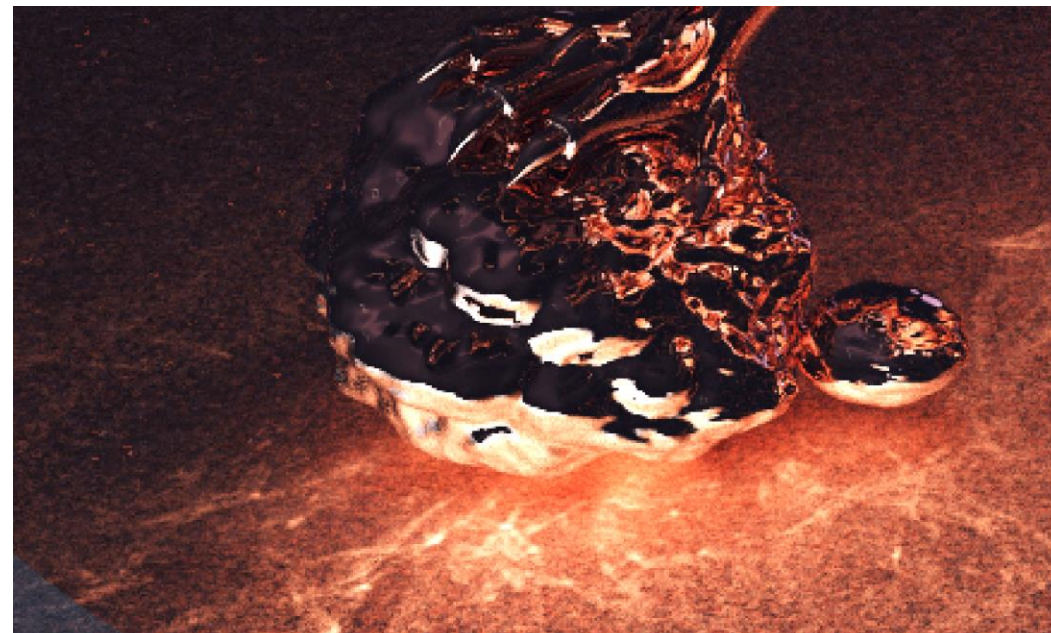
Bias due to spatial relaxation

Rendering sharp caustics is difficult!



Path guiding [Müller et al. 2017]

- Extremely slow to converge

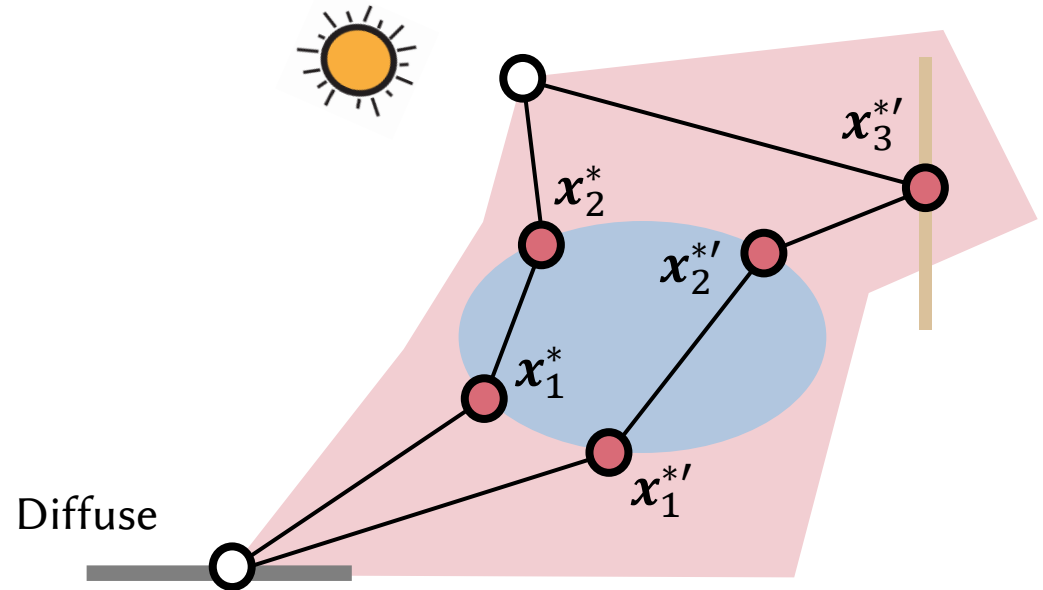
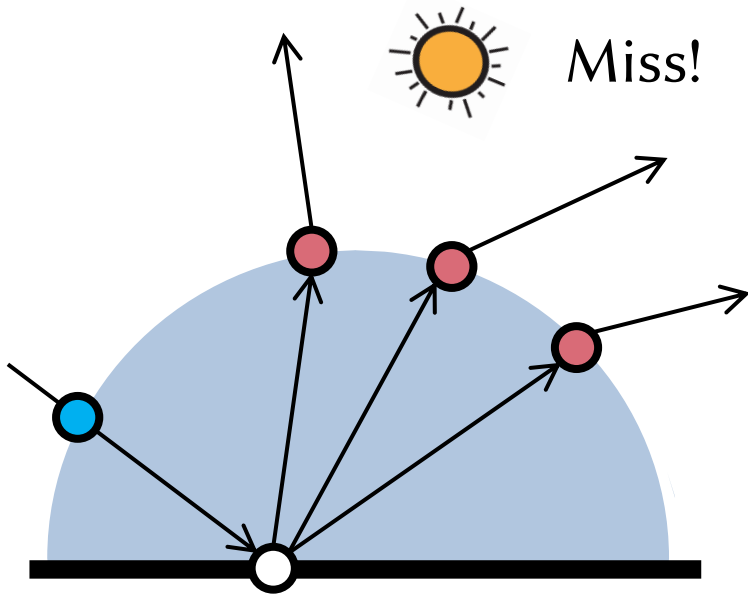


SPPM [Hachisuka et al. 2009]

- Blurring and noise

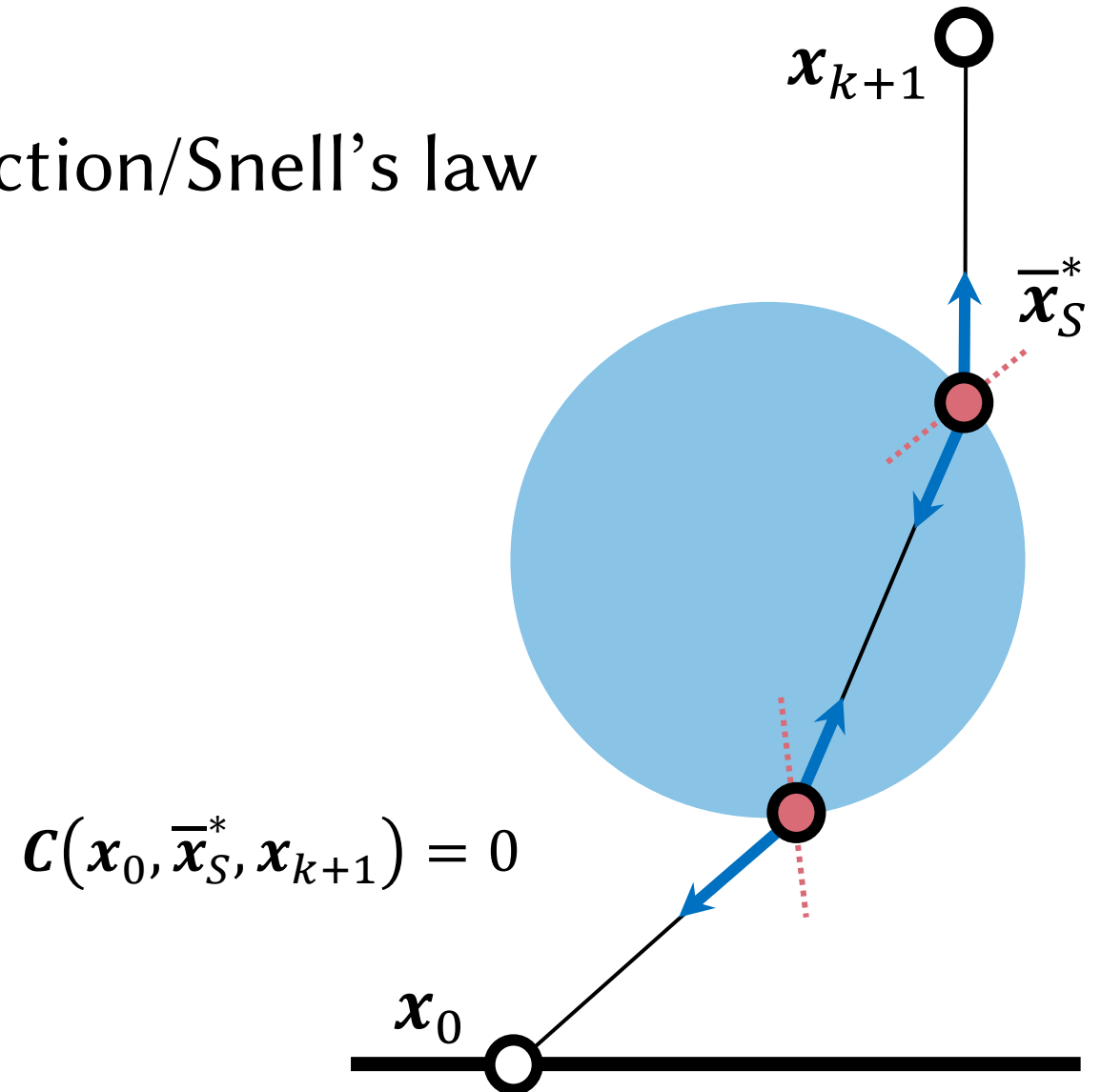
Specular path

- Local sampling fails to reach a (near-)point light source
- Specialized methods connect endpoints with specular vertices
- **Problem: How to connect?**

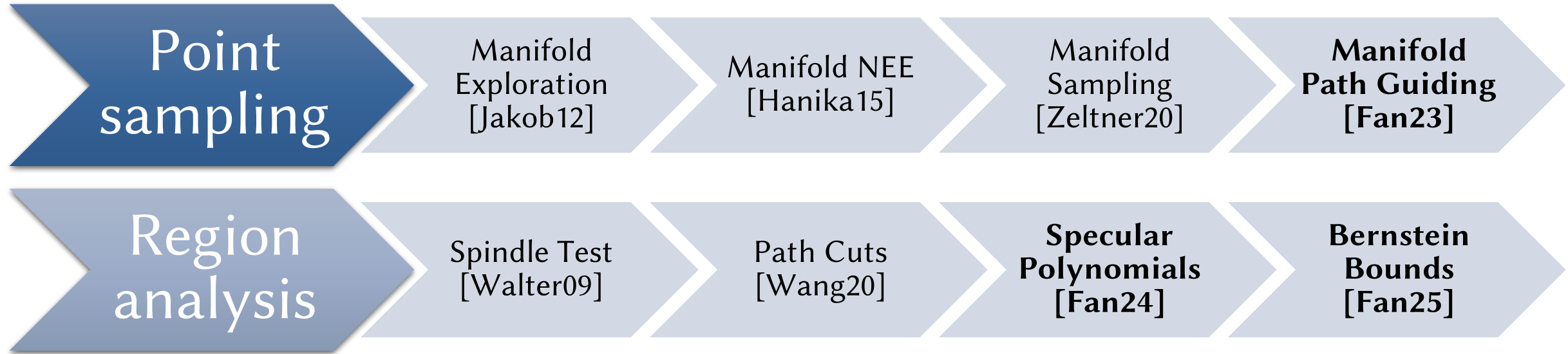


Specular constraints

- Specular vertices satisfy the reflection/Snell's law
- Just solve the equations!
 - k specular vertices
 - $2k$ variables
 - $2k$ equations
- Difficult to solve!



Recent advancements



Other works

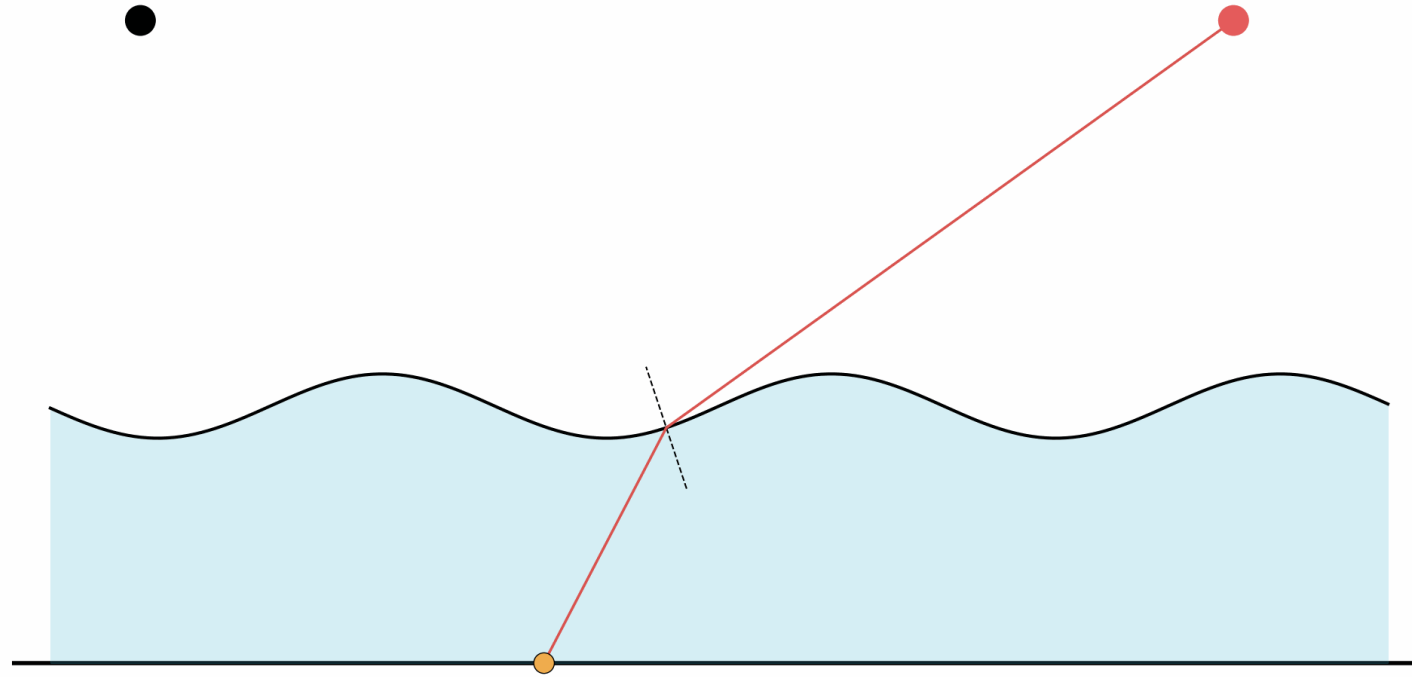
Analytic GGX integral [Loubet20], Path-cut-based guiding [Li22]

Large jump for manifold sampling [Jhang22], Specular path reuse [Xu23], Neural manifold sampling [Yu23]

Photon-guided sampling [Lee24], Single-bounce dimension reduction [Granizo-Hidalgo24], Position-normal manifold clustering [Wu25]

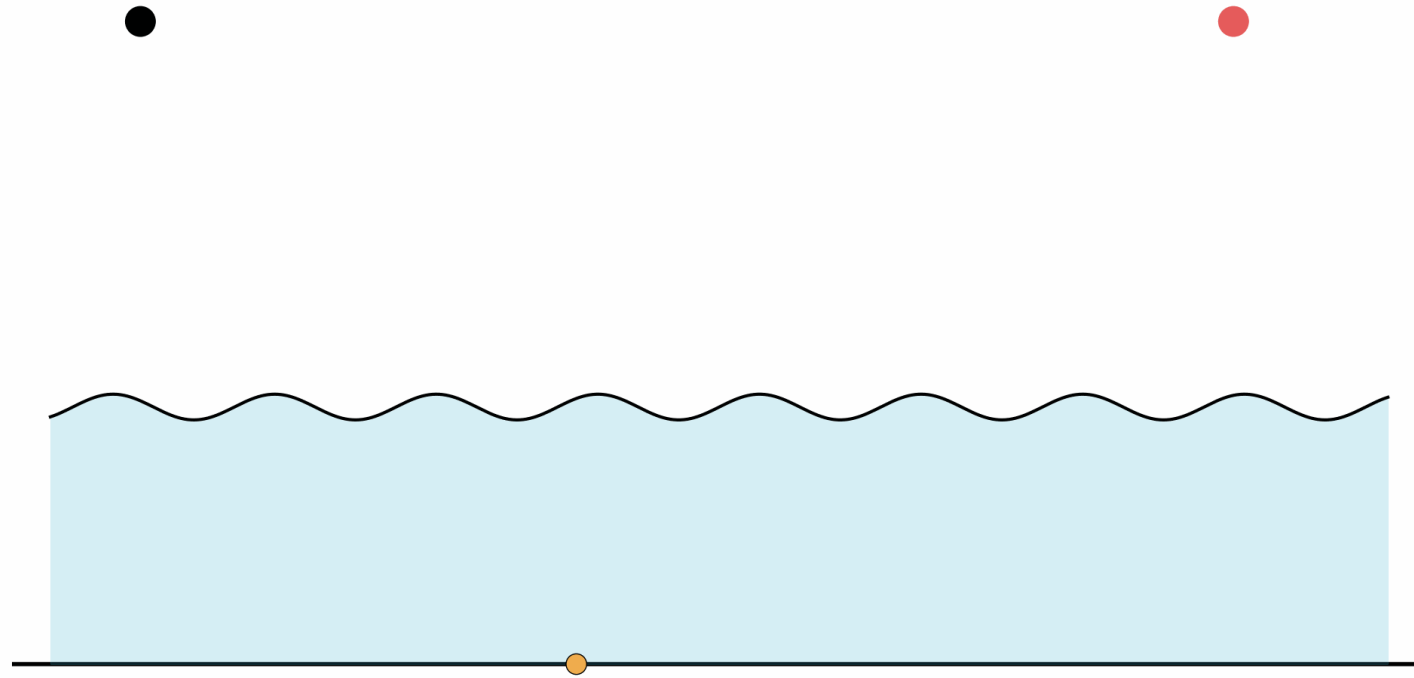
Manifold: deterministic init.

- Manifold next event estimation [Hanika et al. 2015]



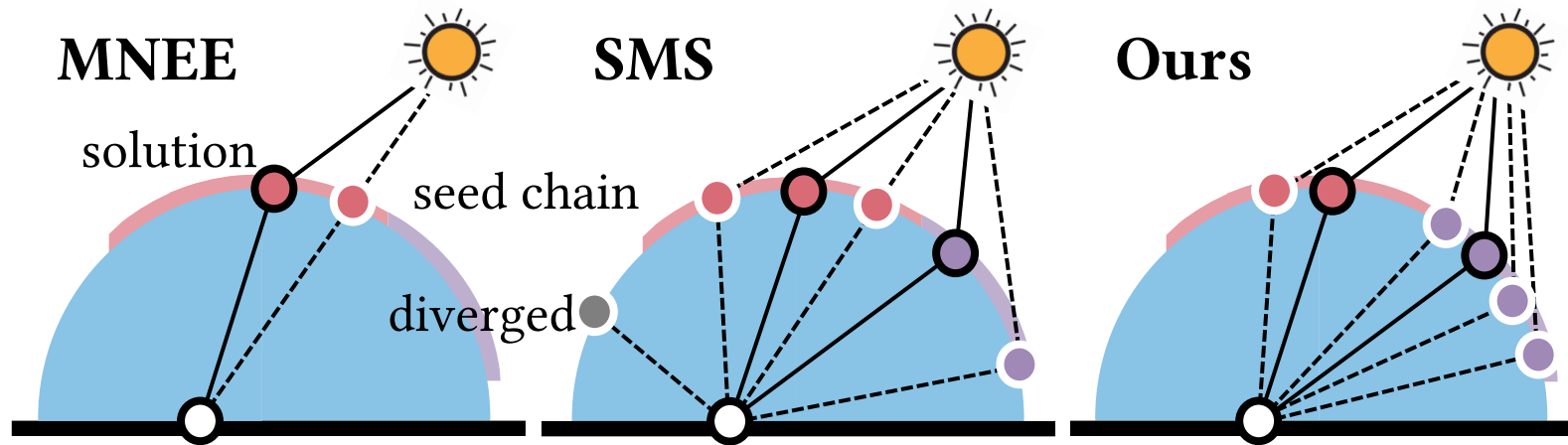
Manifold: uniform stochastic sampling

- Specular manifold sampling [Zeltner et al. 2020]



Manifold path guiding: motivation

- MNEE finds **at most one solution**, resulting in **energy loss**
- SMS **uniformly** samples the seeds, which leads to **high variance**
- **Goal:** find paths that are not only **admissible** but also “**important**”



Fitting

Data-driven Modeling

Manifold Path Guiding for Importance Sampling Specular Chains

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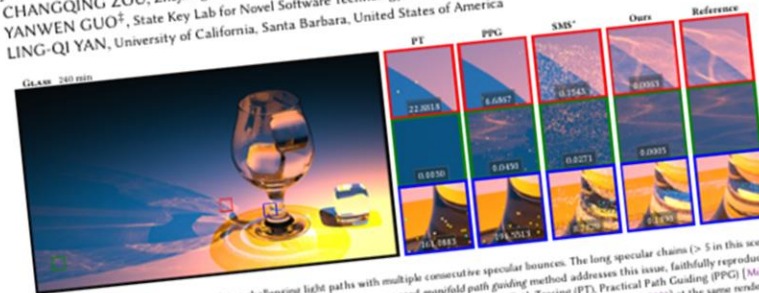


Fig. 1. Rendering a complex scene involving challenging light paths with multiple conservative specular bounces. The long specular chains (> 5 in this scene) create great obstacles to existing path sampling algorithms, while the proposed manifold path guiding method addresses this issue, faithfully reproducing high-frequency caustics and noticeably reducing the variance. Here, we compare our method with Path Tracing (PT), Practical Path Guiding (PPG) [Müller 2019; Müller et al. 2017], and an extension [supporting various chain types] of Specular Manifold Sampling (SMS*) [Zellner et al. 2020] at the same rendering time. Quantitative error in terms of MSE is reported for each clump.

time. Quantitative error in terms of MSE is reported for each path.

least first authors.

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generated in the context of importance sampling and converge to admissible chains through manifold walks. We reach the goal of importance sampled chains as the continuous space verifies that those observations summed chain the discrete admissible specular chains. Based on these observations we propose a new importance sampling scheme, named path guided, by replacing the discrete admissible specular chains. This new scheme is based on sampling theoretical analyses, a progressive scheme, manifold path guiding, is designed and implemented to importance sample challenging paths features. The results demonstrate that our method can significantly reduce variance in regular long specular chains. To our best knowledge, this is the first general framework for importance sampling discrete specular chains in regular long specular chains. Our experimental results demonstrate that our method Monte Carlo simulation. Extensive experiments with up to 60° variance performance state-of-the-art unbiased samplers demonstrate that our method outperforms, especially in typical scenes containing long specular chains and complex visibility.

— Ray tracing

CCS Concepts • **Computing methodologies** → Ray tracing,

CCS Concepts: • **Computing methodologies** → **Ray tracing**
Additional Key Words and Phrases: Specular chain, importance sampling, Caustics

ACM Reference Format:
Zhaomin Fan, Pengfei Hong, Jie Guo, Changqing Zou, Yanwen Guo, and Ling-Qi Yan. 2023. Manifold Path Guiding for Importance Sampling Spectral Chains. *ACM Trans. Graph.* 42, 6, Article 1 (December 2023), 14 pages. <https://doi.org/10.1145/3618360>

1 INTRODUCTION

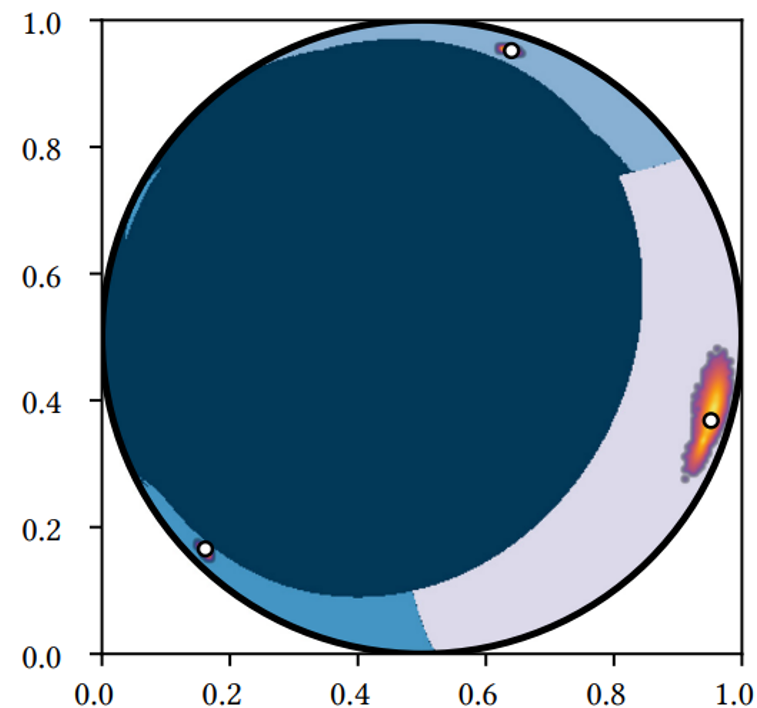
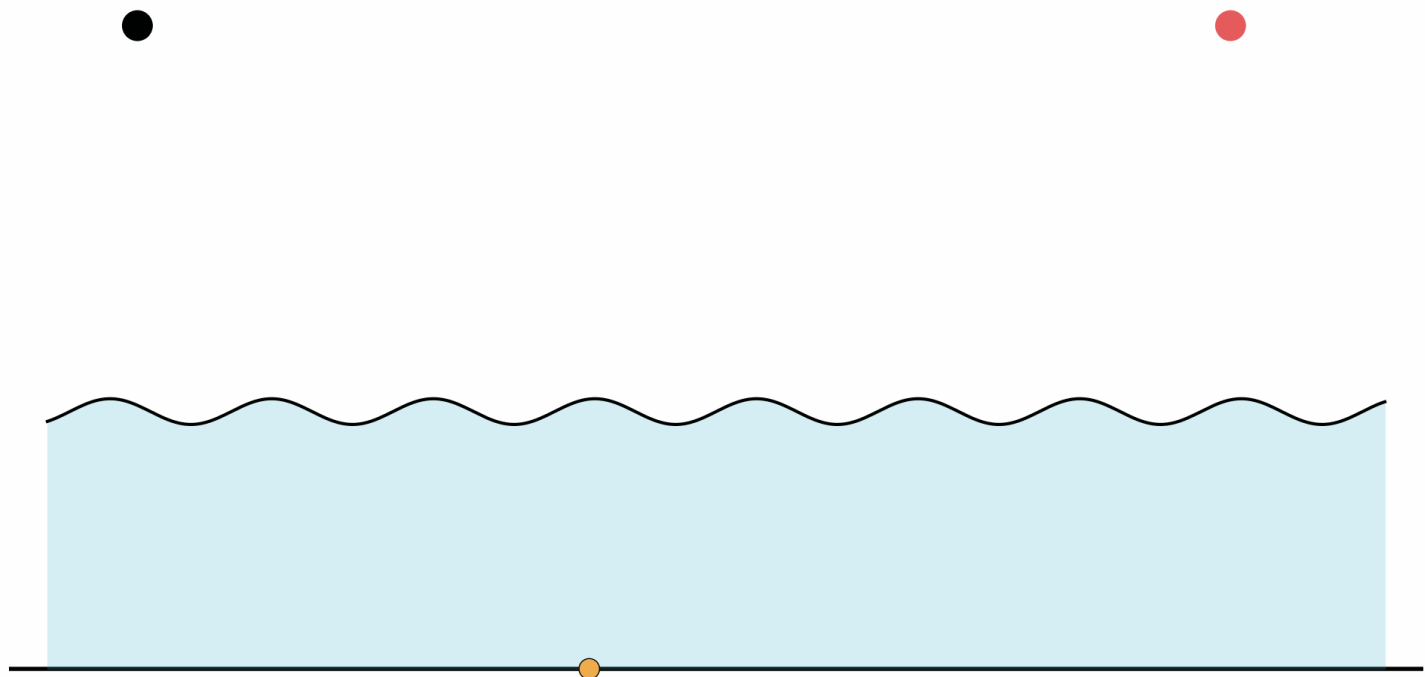
1 INTRODUCTION

Monte Carlo (MC) integration using stochastic samples has long been the de facto solution to the problem of physically-based light transport simulation [Christensen and Jarosz 2016; Fiascone et al. 2018; Keller et al. 2015]. Over the past decades, great efforts have

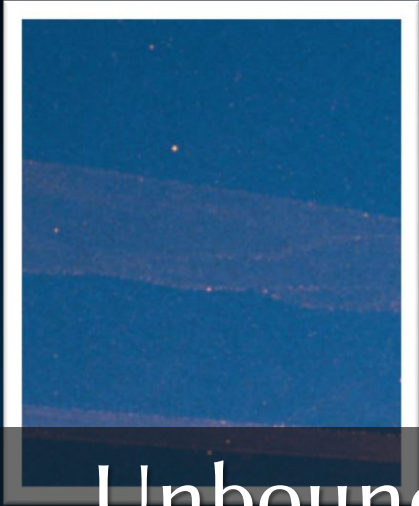
ACM Trans. Graph., Vol. 42, No. 6, Article 1. Publication date: December 2023.

Manifold: importance sampling

- Manifold path guiding [Fan et al. 2023]



Results (4 hours)



Unbounded variance / Newton solver's local convergence



Solving

Analytic Modeling

Specular Polynomials

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Fig. 1. We render a shop window scene featuring challenging caustics and complex visibility, using our pipeline based on specular polynomials. The caustics stem from colored point light sources placed inside a dielectric object, and the whole scene is viewed through a transparent window. Such a configuration makes most existing rendering algorithms fail, while our method succeeds in reproducing the stunning light transport effect. The insets show equal time (10 min) comparisons against Stochastic Progressive Photon Mapping (SPPM) [Hachisuka and Jensen 2009] and Manifold Path Guiding (MPG) [Fan et al. 2023].

Finding valid light paths that involve specular vertices in Monte Carlo rendering requires solving many non-linear, transcendental equations in

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high-dimensional space. Existing approaches heavily rely on Newton iterations in path space, which are limited to obtaining at most a single solution each time and easily diverge when initialized with improper seeds.

We propose specular polynomials, a Newton iteration-free methodology for finding a complete set of admissible specular paths connecting two sub-triangles in a scene. The core is a reformulation of specular constraints into polynomial systems, which makes it possible to reduce the task to a univariate root-finding problem. We first derive bivariate systems utilizing rational coordinate mapping between the coordinates of consecutive vertices. Subsequently, we adopt the hidden variable resultant method for variable elimination, converting the problem into finding zeros of the determinant of univariate matrix polynomials. This can be effectively solved through Lagrange expansion for one bounce and a bisection solver for more bounces.

Our solution is generic, completely deterministic, accurate for the case of one bounce, and GPU-friendly. We develop efficient CPU and GPU implementations and apply them to challenging glints and caustic rendering. Experiments on various scenarios demonstrate the superiority of specular polynomial-based solutions compared to Newton iteration-based counterparts. Our implementation is available at <https://github.com/mellon/spoly>.

CCS Concepts • Computing methodologies → Ray tracing.

ACM Trans. Graph., Vol. 43, No. 4, Article 1. Publication date: August 2024.

Why Polynomials?

- Root-solving with **global** convergence
 - **Root isolation** by computing derivatives recursively
 - **Eigenvalue**: QR/QZ decomposition
- **Key idea**: derive (1D) polynomial formulations

Specular Polynomials

- Multivariate polynomial constraints $F(\mathbf{u}_{i-1}, \mathbf{u}_i, \mathbf{u}_{i+1}) = 0$

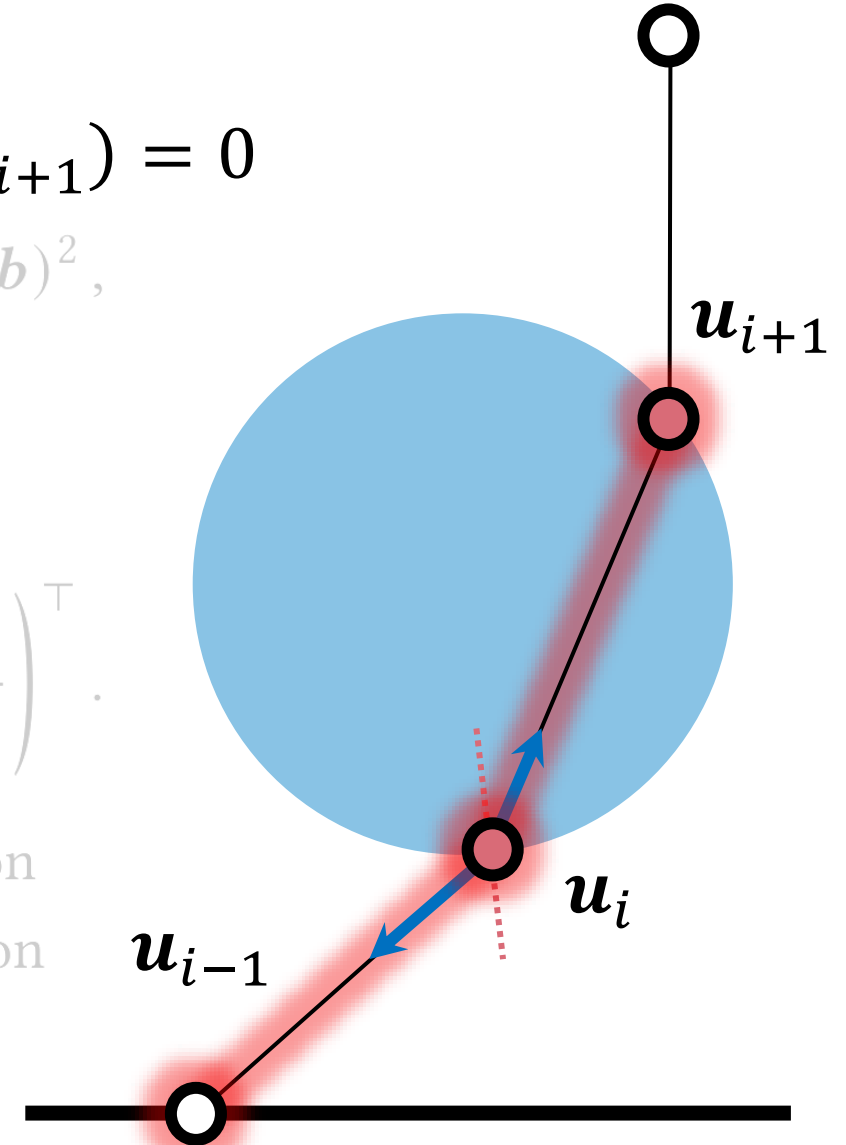
$$\begin{cases} F = \mathbf{d}_{k-1}^2 ((\mathbf{d}_{k-2} \times \mathbf{n}_{k-1}) \cdot \mathbf{b})^2 - \eta^2 \mathbf{d}_{k-2}^2 ((\mathbf{d}_{k-1} \times \mathbf{n}_{k-1}) \cdot \mathbf{b})^2, \\ G = (\mathbf{d}_{k-2} \times \mathbf{n}_{k-1}) \cdot \mathbf{d}_{k-1}. \end{cases}$$

- Rational coordinate mapping $\mathbf{u}_{i+1} = f(\mathbf{u}_i, \mathbf{u}_{i-1})$

$$\mathbf{u}_{i+1} = \left(\frac{(\tilde{\mathbf{d}}_i \times \mathbf{e}_{i+1,2}) \cdot (\mathbf{x}_i - \mathbf{p}_{i+1,0})}{(\tilde{\mathbf{d}}_i \times \mathbf{e}_{i+1,2}) \cdot \mathbf{e}_{i+1,1}}, \frac{((\mathbf{x}_i - \mathbf{p}_{i+1,0}) \times \mathbf{e}_{i+1,1}) \cdot \tilde{\mathbf{d}}_i}{(\tilde{\mathbf{d}}_i \times \mathbf{e}_{i+1,2}) \cdot \mathbf{e}_{i+1,1}} \right)^\top.$$

$$\tilde{\mathbf{d}}_i = \begin{cases} (\mathbf{n}_i \cdot \mathbf{n}_i) \mathbf{d}_{i-1} - 2(\mathbf{d}_{i-1} \cdot \mathbf{n}_i) \mathbf{n}_i, & \text{for reflection} \\ \eta'_i ((\mathbf{n}_i \cdot \mathbf{n}_i) \mathbf{d}_{i-1} - (\mathbf{d}_{i-1} \cdot \mathbf{n}_i) \mathbf{n}_i) - \sqrt{\beta_i} \mathbf{n}_i, & \text{for refraction} \end{cases}$$

$$\beta_i = (1 - \eta_i'^2)(\mathbf{n}_i \cdot \mathbf{n}_i)(\mathbf{d}_{i-1} \cdot \mathbf{d}_{i-1}) + \eta_i'^2(\mathbf{d}_{i-1} \cdot \mathbf{n}_i)^2.$$



Resultant elimination for two polynomials

- The necessary condition of
 - the existence of common roots of
 - **two** polynomials
- Bézout's resultant

$$r(v_1) = \det \mathbf{R}(v_1)$$



$$R_{i,j} = \sum_{k=0}^{\min(i, n-1-j)} \left(a_{i-k}(v_1) b_{j+1+k}(v_1) - b_{i-k}(v_1) a_{j+1+k}(v_1) \right)$$

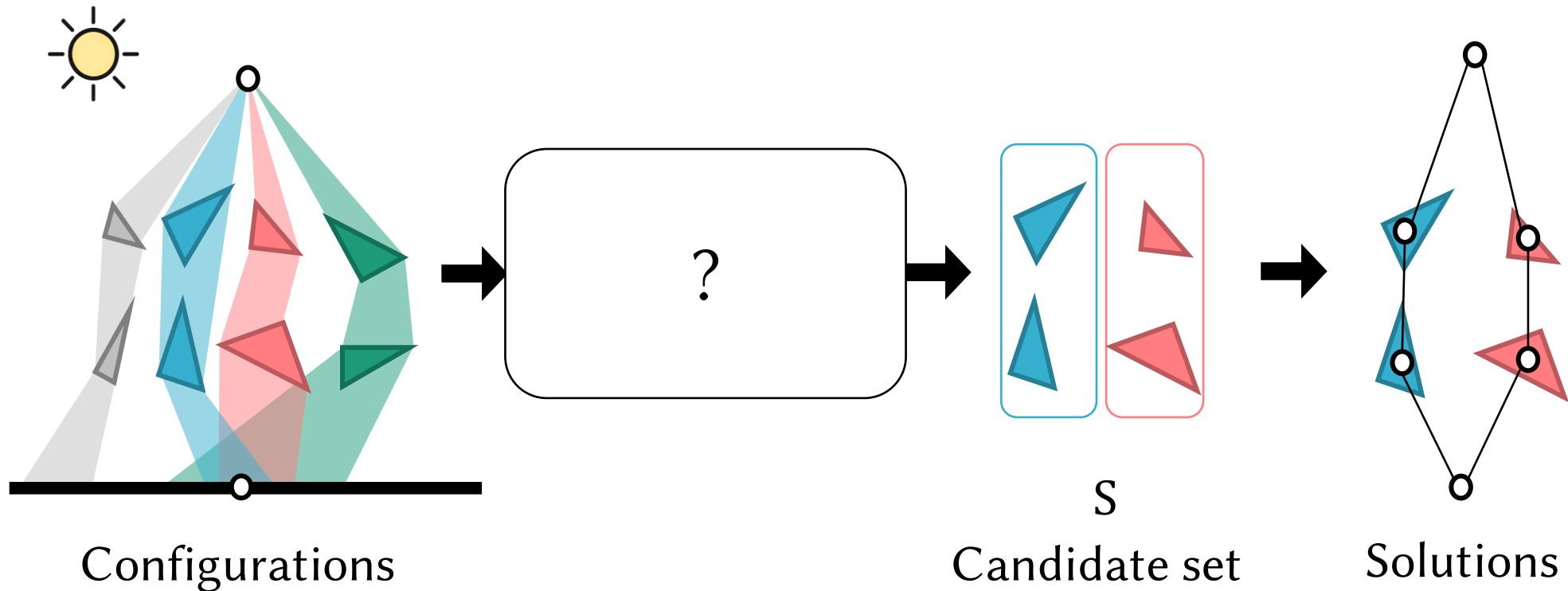
$$\begin{cases} a(u_1, v_1) = \sum_{i=0}^n a_i(v_1) u_1^i = 0, \\ b(u_1, v_1) = \sum_{i=0}^n b_i(v_1) u_1^i = 0. \end{cases}$$

Only in v_1 now!

Deterministic search for specular paths

- Select triangle tuples first (focus in Fan et al. 2025, also using SP)
- Then find solutions using Newton's method or polynomial solvers

[Walter et al. 2009] [Wang et al. 2020] [Fan et al. 2024]



Bounding

Analytic Modeling

Bernstein Bounds for Caustics

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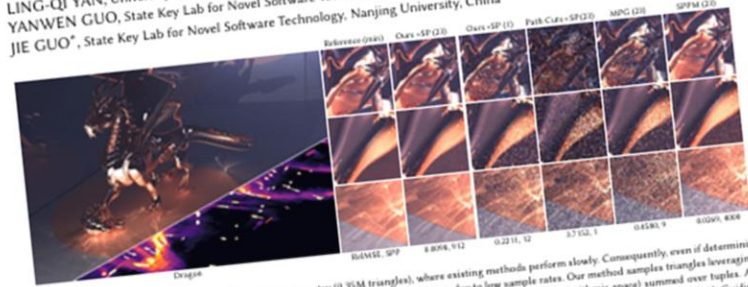


Fig. 1. Rendering sharp caustics reflected by complex geometry (0.35M triangles), where existing methods perform slowly. Consequently, even if deterministically searching for the complete set of admissible paths, they still produce high variance due to low sample rates. Our method samples triangles leveraging the bounds for caustics, leading to more converged results. We visualize the irradiance bound (in the base 10 logarithmic space) summed over tuples. All methods render single reflections only. We compare with Path Cuts [Wang et al. 2020], Specular Polynomials (SP) [Fan et al. 2024], Manifold Path Guiding (MPG) [Fan et al. 2021], and Stochastic Progressive Photon Mapping (SPPM) [Hachisuka and Jensen 2009]. Two budgets for ours focus on equal time (32 sec for precomputation with finer subdivisions, 21 min in total) and roughly equal quality comparisons (9 sec for precomputation, 1 min in total), respectively.

Systematically simulating specular light transport requires an exhaustive search for triangle tuples containing admissible paths. Given the extreme inefficiency of enumerating all combinations, we significantly reduce the search domain by stochastically sampling such tuples. The challenge is to design proper sampling probabilities that keep the noise level controllable. Our key insight is that by bounding the irradiance contributed by each triangle tuple at a given position, we can sample a subset of triangle tuples with potentially high contributions. Although low-contribution tuples are assigned a negligible probability, the overall variance remains low.

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Therefore, we derive position and irradiance bounds for caustics contributed by each triangle tuple, introducing a bounding property of rational functions on a Bernstein basis. When formulating position and irradiance expressions into rational functions, we handle non-rational parts through re-parameterizations to maintain bounding validity. Finally, we carefully design the sampling probabilities by optimizing the upper bound of the variance, expressed only using the position and irradiance bounds.

The bound-driven sampling of triangle tuples is intrinsically unbiased even without defensive sampling. It can be combined with various unbiased and biased root-finding techniques within a local triangle domain. Extensive evaluations show that our method enables the fast and reliable rendering of complex caustics efficiently. Yet, our method is efficient for no more than two specular vertices, where complexity grows sublinearly to the number of triangles and linearly to that of emitters, and does not consider the Fresnel and visibility terms. We also rely on parameters to control subdivisions. The implementation is available at <https://github.com/mofan/bound-caustics>.

CCS Concepts • Computing methodologies → Ray tracing

Additional Key Words and Phrases: specular, caustics

ACM Reference Format:

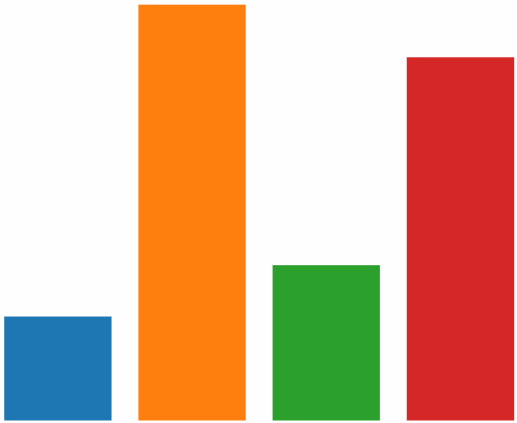
Zhimin Fan, Chen Wang, Yiming Wang, Boxuan Li, Yuxuan Guo, Ling-Qi Yan, Yanwen Guo, and Jie Guo. 2025. Bernstein Bounds for Caustics. ACM Trans. Graph. 44, 4, Article 1 (August 2025), 15 pages. <https://doi.org/10.1145/3771145>

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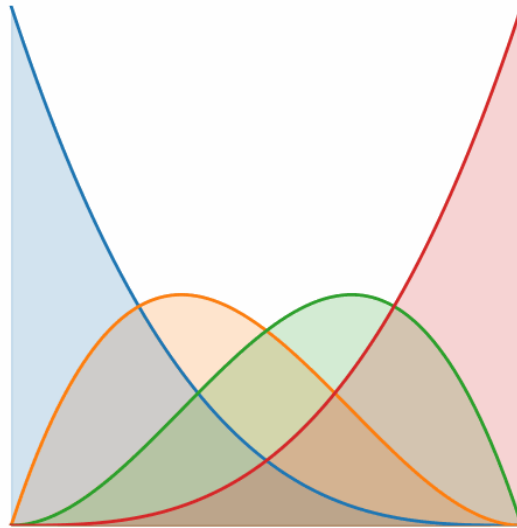
Bernstein polynomial basis

$$f(x) = \sum_{i=0}^n b_i^f x^i (1-x)^{n-i}$$

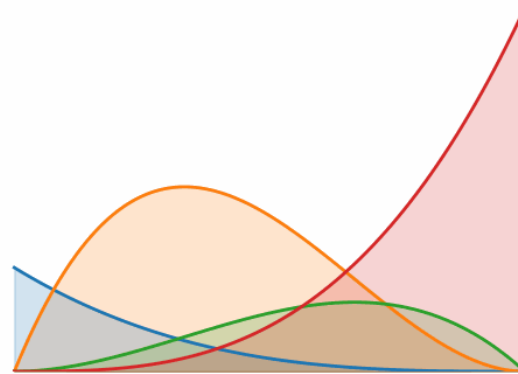
Coefficients



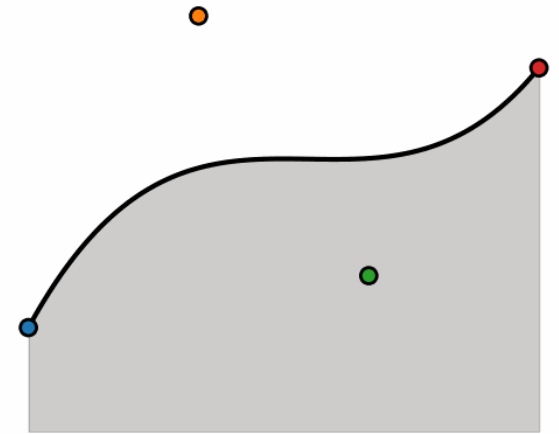
Basis



Components



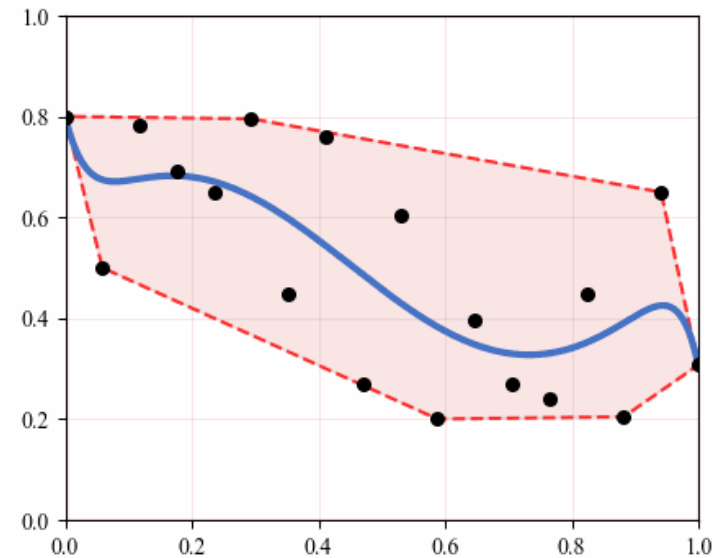
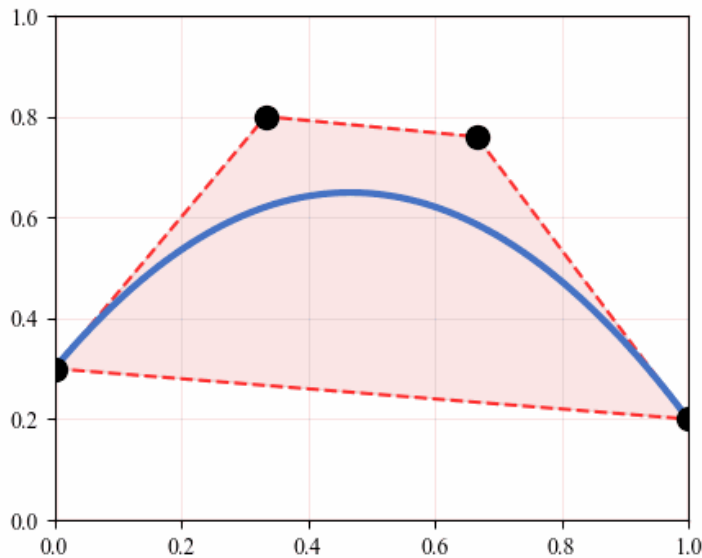
Sum



Bounding the range of polynomials

$$f(x) = \sum_{i=0}^n b_i^f x^i (1-x)^{n-i}$$

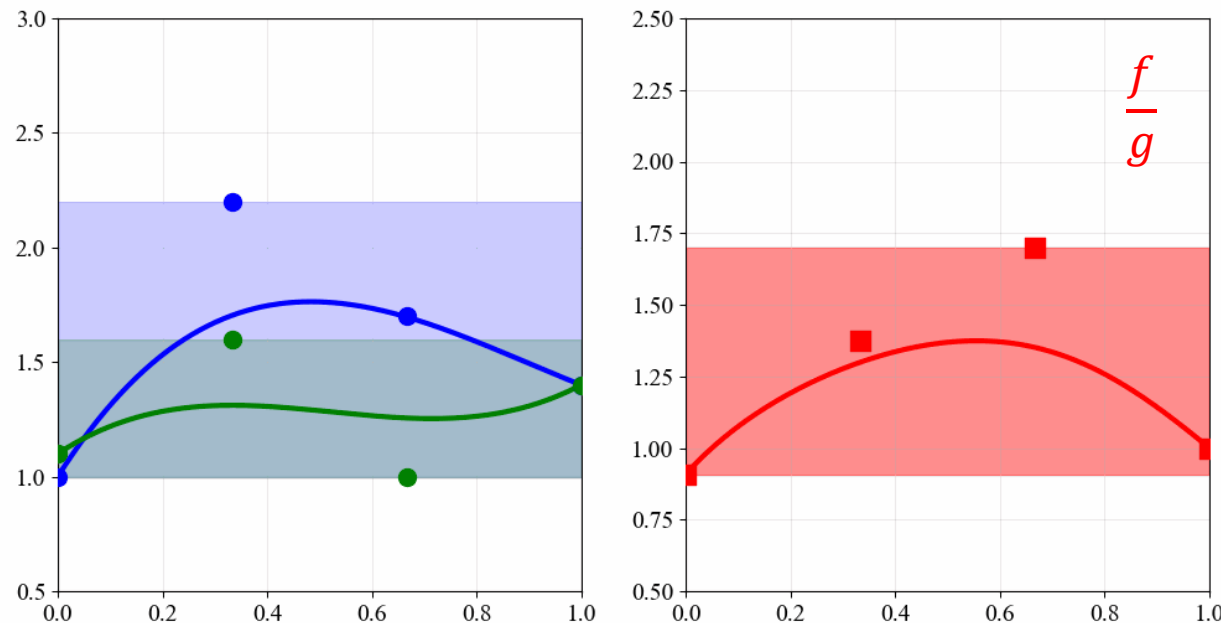
- $(x, f(x))$ falls in the **convex hull** formed by $\left\{ \left(\frac{i}{n-1}, b_i^f \right) \mid 0 \leq i \leq n \right\}$



Bounding the range of rational functions

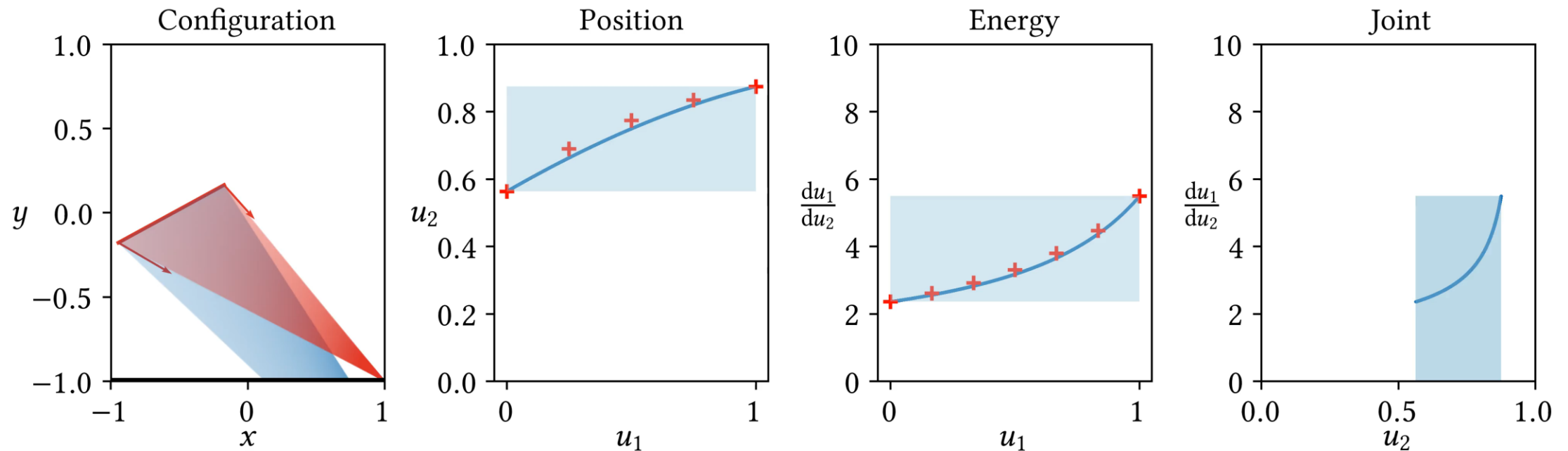
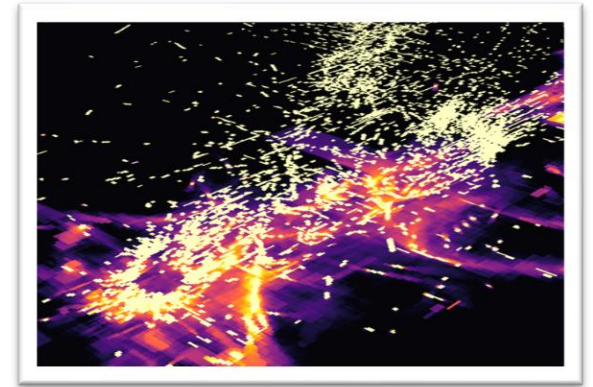
$$\min_{i=0}^n \frac{b_i^f}{b_i^g} \leq \frac{f(x)}{g(x)} \leq \max_{i=0}^n \frac{b_i^f}{b_i^g} \quad (g(x) \neq 0)$$

[Narkawicz et al. 2012]



Bernstein bounds for caustics

- Represent caustics with Bernstein polynomials
- Obtain their bounds using the bounding property



Summary

- **Fitting distributions for the constrain solutions (23')**

- General
- *Needs training/initial distributions, suffers from outlier cases, so...*

- **Reformulating constraints into polynomials (24')**

- Thus enabling many mathematical tools
- **Solving (24')**
 - Resultant elimination | Root isolation/QZ decomposition
 - *Too slow due to enumeration, so...*

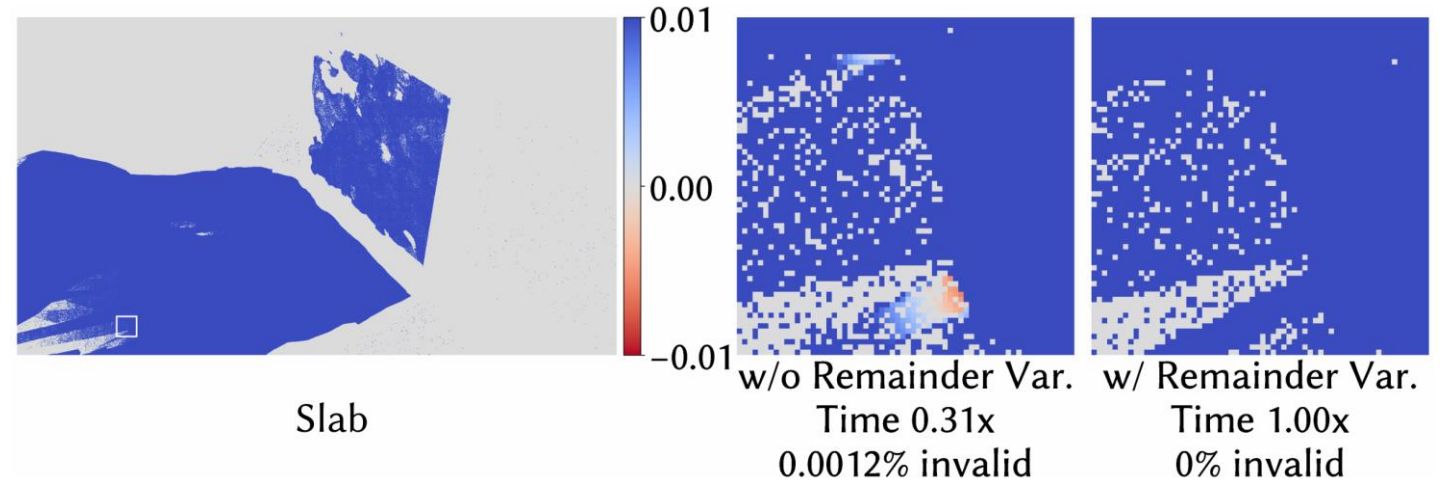
- **Bounding (25')**

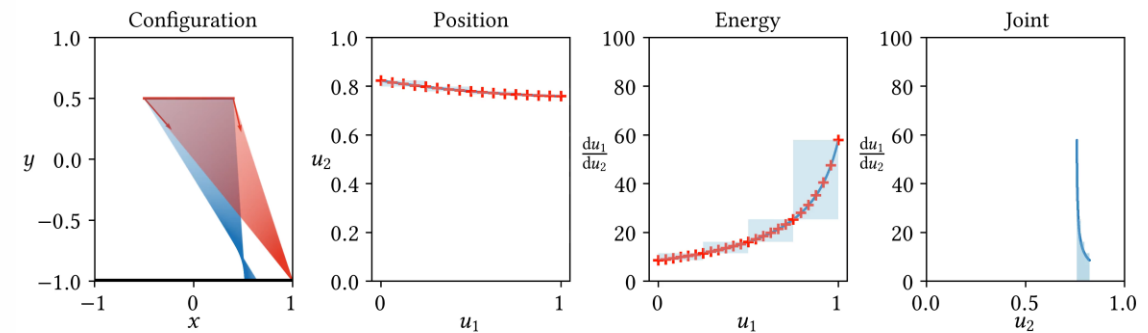
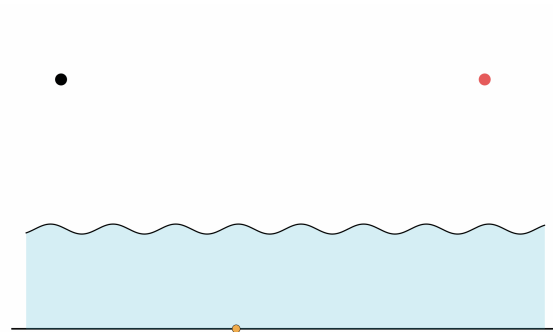
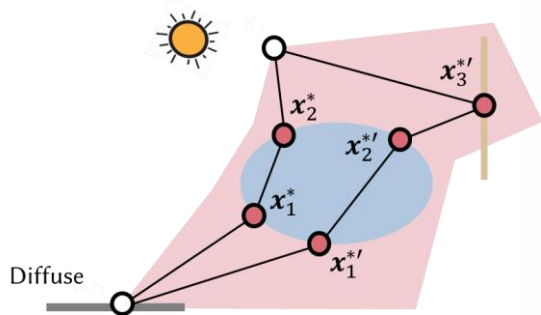
- Bernstein polynomials | Regression with remainders
- *Producing an approximated distribution of the constrain solutions*

Open questions

Performance

- **We focus on**
 - a theoretically sound solution
 - and accurate modeling
- **Allow violations/approximations** to be more efficient
 - How to strike a balance?





$$\begin{cases} F = d_{k-1}^2 ((d_{k-2} \times n_{k-1}) \cdot b)^2 - \eta^2 d_{k-2}^2 ((d_{k-1} \times n_{k-1}) \cdot b)^2, \\ G = (d_{k-2} \times n_{k-1}) \cdot d_{k-1}. \end{cases}$$

$$u_{i+1} = \left(\frac{(\tilde{d}_i \times e_{i+1,2}) \cdot (x_i - p_{i+1,0})}{(\tilde{d}_i \times e_{i+1,2}) \cdot e_{i+1,1}}, \frac{((x_i - p_{i+1,0}) \times e_{i+1,1}) \cdot \tilde{d}_i}{(\tilde{d}_i \times e_{i+1,2}) \cdot e_{i+1,1}} \right)^\top.$$

$$\tilde{d}_i = \begin{cases} (n_i \cdot n_i) d_{i-1} - 2(d_{i-1} \cdot n_i) n_i, & \text{for reflection} \\ \eta'_i ((n_i \cdot n_i) d_{i-1} - (d_{i-1} \cdot n_i) n_i) - \sqrt{\beta_i} n_i, & \text{for refraction} \end{cases}$$

$$\beta_i = (1 - \eta_i'^2)(n_i \cdot n_i)(d_{i-1} \cdot d_{i-1}) + \eta_i'^2(d_{i-1} \cdot n_i)^2.$$

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Thank You!

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Contact me if you have any questions!