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THE PREMIER CONFERENCE & EXHIBITION ON COMPUTER GRAPHICS & INTERACTIVE TECHNIQUES

Specular Polynomials

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Why Polynomials?

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Root-solving with **global** convergence

(QR/QZ, **root isolation**)

Why Polynomials?

Solve $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$

$$p'(x) = \qquad a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

The roots of p' determines the monotonic pieces of p

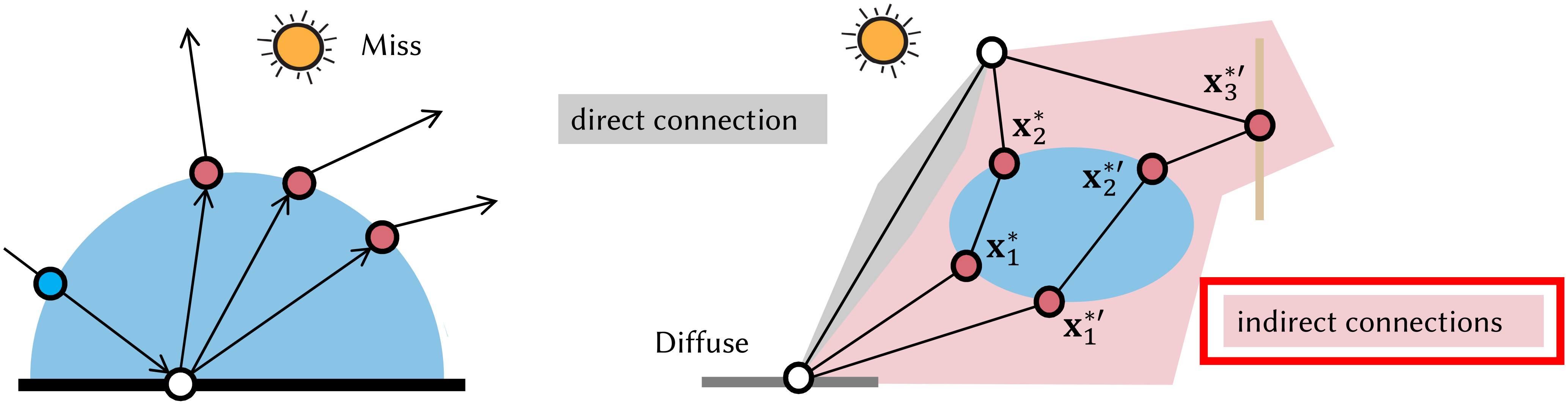
Recursion + bisection on each piece

Specular Polynomials

A New **Solver** for Specular Paths

Specular path sampling

- Local sampling failed to reach a (near-)point emitter
- Specialized methods connect **endpoints** with **specular vertices**
- How to connect?



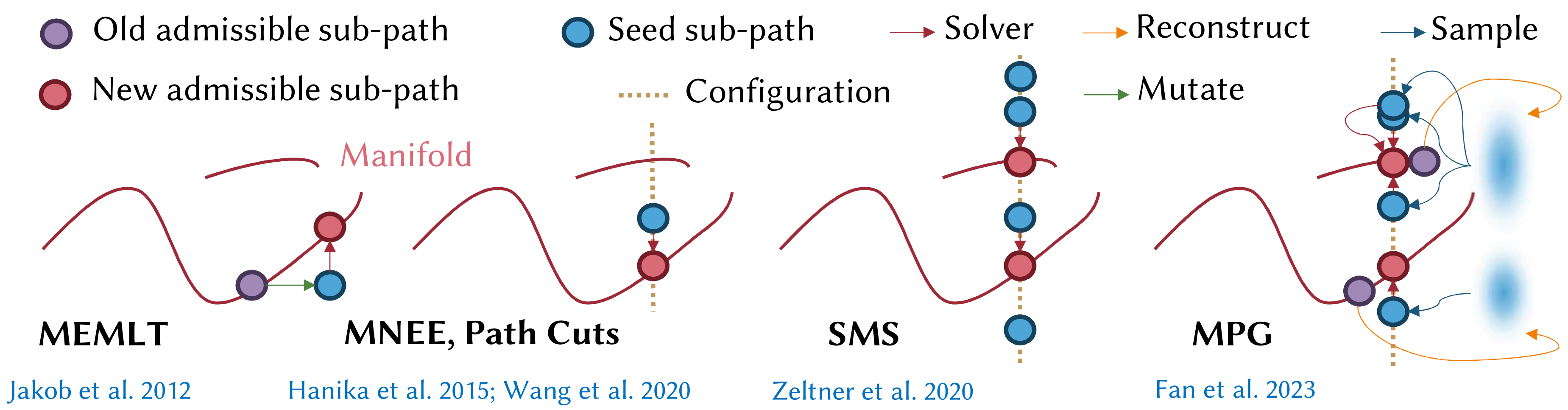
Specular constraints

- Specular chains satisfy the reflection/Snell's law
- Finding specular chains connecting two endpoints requires solving the equations

- k vertices
- $2k$ variables (parameterized via barycentric coordinates)
- $2k$ independent equations

Prior works

- Solve the constraint equation via **Newton's method**
  - with meticulously chosen seeds
- Suffers from **unbounded convergence**
  - extremely high variance (fireflies) or bias (energy loss)



Our contributions

- **A polynomial formulation of specular constraints**
  - derived by combining vertex constraint polynomials and
  - rational coordinate mappings between barycentric coordinates
- **A specular path solver using hidden variable resultant method**
  - combined with direct or eigenvalue solvers
  - which is deterministic and free from multivariate Newton iterations
- **Applications to glints and caustics rendering**
  - which achieves fast and noise-free rendering of specular light transport effects

Laws → MVP

Reflection/Snell's laws → Multivariate Polynomials

Problem setup

$$\boldsymbol{u}_i = (1 - u_i - v_i, u_i, v_i)^\top$$

$$\boldsymbol{x}_i = (1 - u_i - v_i)\boldsymbol{p}_{i,0} + u_i\boldsymbol{p}_{i,1} + v_i\boldsymbol{p}_{i,2} = \boldsymbol{P}_i\boldsymbol{u}_i.$$

$$\hat{\boldsymbol{n}}_i = \frac{\boldsymbol{n}_i}{\|\boldsymbol{n}_i\|}, \quad \boldsymbol{n}_i = (1 - u_i - v_i)\boldsymbol{n}_{i,0} + u_i\boldsymbol{n}_{i,1} + v_i\boldsymbol{n}_{i,2} = \boldsymbol{N}_i\boldsymbol{u}_i.$$

Symbol	Description
$\boldsymbol{x}_0, \boldsymbol{x}_{k+1}$	Position of non-specular separators
$\boldsymbol{x}_i$	Position of specular vertices
$\boldsymbol{P}_i$	Position matrix $(\boldsymbol{p}_{i,0}, \boldsymbol{p}_{i,1}, \boldsymbol{p}_{i,2})$
$\boldsymbol{e}_{i,1}, \boldsymbol{e}_{i,2}$	Vector of triangle edges
$\boldsymbol{n}_i$	Un-normalized linearly interpolated normal of $\boldsymbol{x}_i$
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$\boldsymbol{t}_{i,1}, \boldsymbol{t}_{i,2}$	Tangent vectors of $\boldsymbol{x}_i$, computing from $\boldsymbol{n}_i$ and $\boldsymbol{e}_{i,1/2}$
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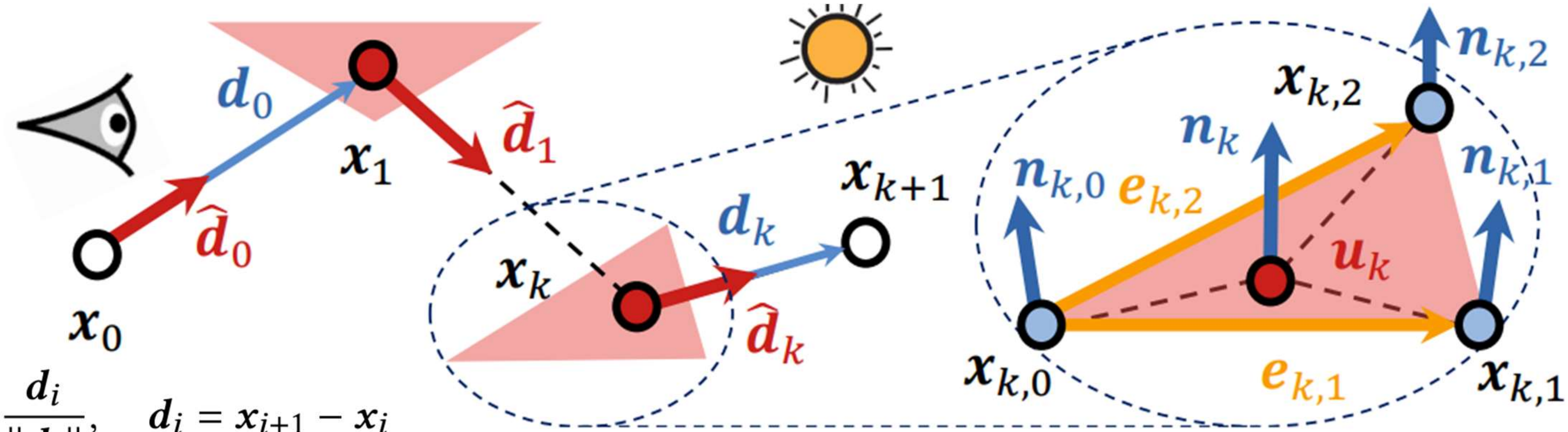
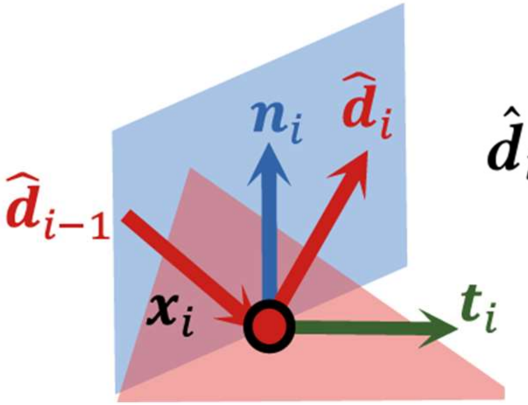


$$\hat{\boldsymbol{d}}_i = \frac{\boldsymbol{d}_i}{\|\boldsymbol{d}_i\|}, \quad \boldsymbol{d}_i = \boldsymbol{x}_{i+1} - \boldsymbol{x}_i$$

Coplanarity constraint



$$\hat{\boldsymbol{d}}_i = \frac{\boldsymbol{d}_i}{\|\boldsymbol{d}_i\|}, \quad \boldsymbol{d}_i = \boldsymbol{x}_{i+1} - \boldsymbol{x}_i$$

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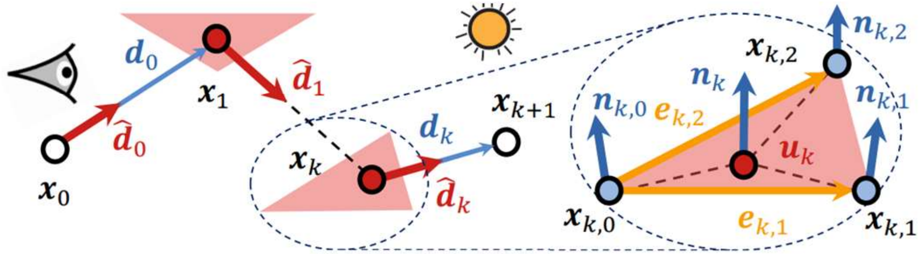
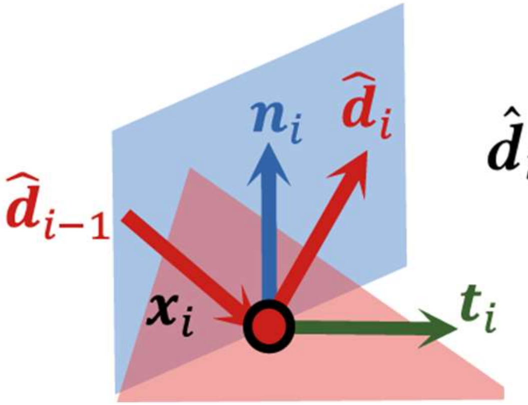


$$\boxed{(\boldsymbol{d}_{i-1} \times \boldsymbol{d}_i) \cdot \boldsymbol{n}_i = 0.}$$

$$\boxed{(\boldsymbol{d}_{i-1} \times (\boldsymbol{x}_{i+1} - \boldsymbol{x}_{i-1})) \cdot \boldsymbol{n}_i = 0.}$$

Angularity constraint



$$\hat{d}_i = \frac{d_i}{\|d_i\|}, \quad d_i = x_{i+1} - x_i$$

Symbol	Description
x_0, x_{k+1}	Position of non-specular separators
x_i	Position of specular vertices
P_i	Position matrix $(p_{i,0}, p_{i,1}, p_{i,2})$
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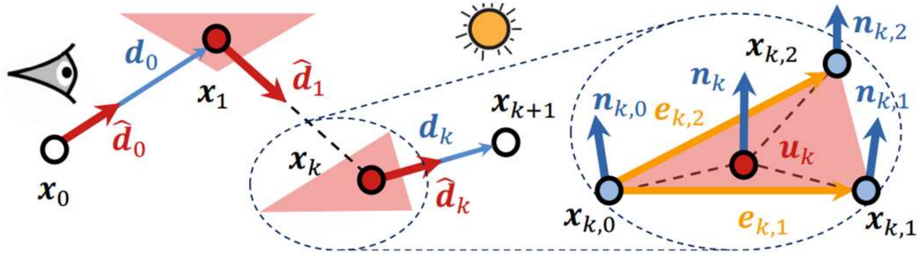

$$\eta_{i-1}\|\hat{d}_{i-1} \times n_i\| = \eta_i\|\hat{d}_i \times n_i\|.$$

$$\Downarrow$$

$$\eta_{i-1}\hat{d}_{i-1} \times n_i = \eta_i\hat{d}_i \times n_i$$

Square roots in denominators!

Angularity: square form

- Projecting onto an arbitrary basis $\boldsymbol{b}$
- Multiplying the common denominator

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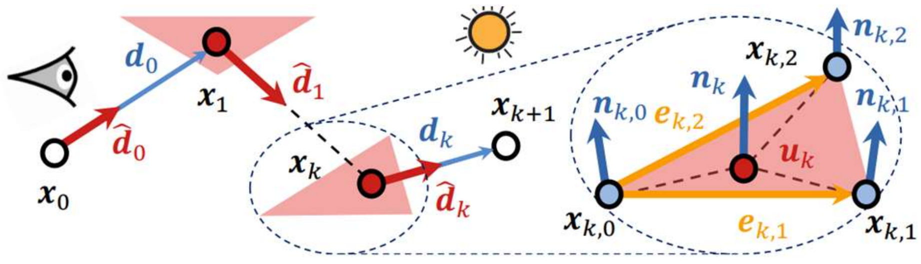

$$\eta_{i-1}\hat{\boldsymbol{d}}_{i-1} \times \boldsymbol{n}_i = \eta_i\hat{\boldsymbol{d}}_i \times \boldsymbol{n}_i$$

$\Downarrow$

$$\boxed{\eta_{i-1}^2\boldsymbol{d}_i^2\left((\boldsymbol{d}_{i-1} \times \boldsymbol{n}_i) \cdot \boldsymbol{b}\right)^2 = \eta_i^2\boldsymbol{d}_{i-1}^2\left((\boldsymbol{d}_i \times \boldsymbol{n}_i) \cdot \boldsymbol{b}\right)^2 .}$$

Angularity: product form

- Exploit the symmetry property
- Reflection only!

$$\hat{\boldsymbol{d}}_{i-1} \cdot \boldsymbol{n}_i = -\hat{\boldsymbol{d}}_i \cdot \boldsymbol{n}_i.$$

$$\hat{\boldsymbol{d}}_i \cdot \boldsymbol{t}_i = \hat{\boldsymbol{d}}_{i-1} \cdot \boldsymbol{t}_i.$$

$$\Downarrow$$

$$\boxed{(\boldsymbol{d}_{i-1} \cdot \boldsymbol{n}_i)(\boldsymbol{d}_i \cdot \boldsymbol{t}_i) + (\boldsymbol{d}_{i-1} \cdot \boldsymbol{t}_i)(\boldsymbol{d}_i \cdot \boldsymbol{n}_i) = 0.}$$

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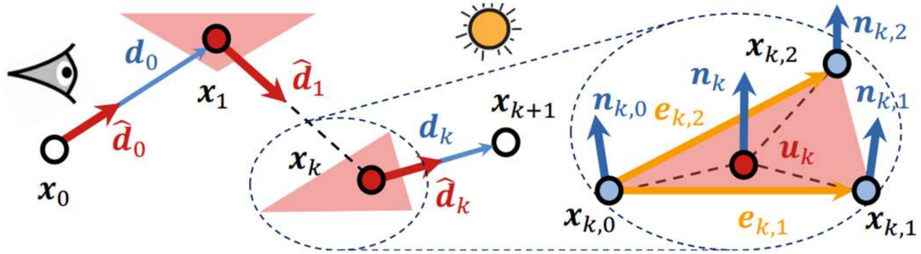

Milestone: Multivariate Specular Polynomials

Constraints	**Formulation I** Equation	Degree I/F	**Formulation II** Equation	Degree I/F
Coplanarity	Consecutive difference form $(\boldsymbol{d}_{i-1} \times \boldsymbol{d}_i) \cdot \boldsymbol{n}_i = 0$	3/2	Endpoint difference form $(\boldsymbol{d}_{i-1} \times (\boldsymbol{x}_{i+1} - \boldsymbol{x}_{i-1})) \cdot \boldsymbol{n}_i = 0$	2/1
Angularity	Square form (for reflection/refraction) $\eta_{i-1}^2 \boldsymbol{d}_i^2 \left((\boldsymbol{d}_{i-1} \times \boldsymbol{n}_i) \cdot \boldsymbol{b}\right)^2 - \eta_i^2 \boldsymbol{d}_{i-1}^2 \left((\boldsymbol{d}_i \times \boldsymbol{n}_i) \cdot \boldsymbol{b}\right)^2 = 0$	6/4	Product form (for reflection) $(\boldsymbol{d}_{i-1} \cdot \boldsymbol{n}_i)(\boldsymbol{d}_i \cdot \boldsymbol{t}_i) + (\boldsymbol{d}_{i-1} \cdot \boldsymbol{t}_i)(\boldsymbol{d}_i \cdot \boldsymbol{n}_i) = 0$	4/2

- Directly solving these multivariate formulations with many variables is not recommended
- In mathematics, the most reliable solvers for polynomial systems are designed for two variables

$$\text{MVP} \rightarrow \text{BVP}$$

2k variables 2 variables

Variable reduction

- Represent $\boldsymbol{u}_i$ with $\boldsymbol{u}_1$
- Why rational?
- To make the final equations polynomial

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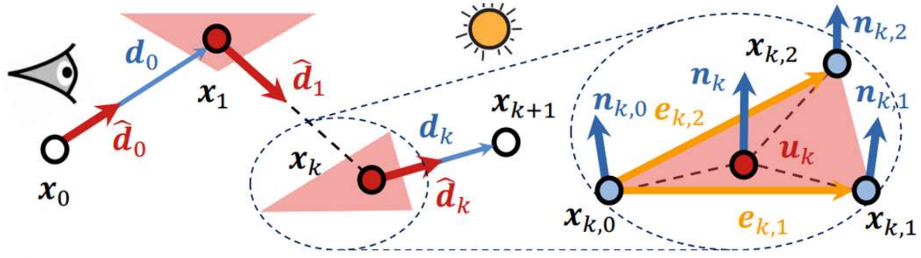

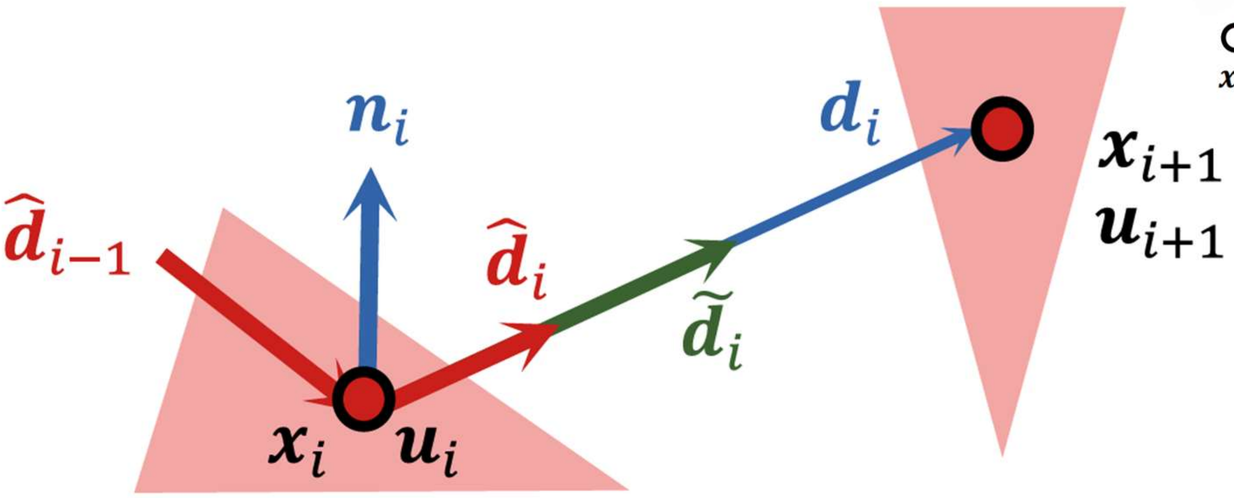

Ray-triangle intersection

$$\boxed{\boldsymbol{u}_{i+1}(\boldsymbol{u}_i, \boldsymbol{u}_{i-1}) = \frac{(\tilde{u}_{i+1}(\boldsymbol{u}_i, \boldsymbol{u}_{i-1}), \tilde{v}_{i+1}(\boldsymbol{u}_i, \boldsymbol{u}_{i-1}))^\top}{\kappa_{i+1}(\boldsymbol{u}_i, \boldsymbol{u}_{i-1})},}$$

$$\tilde{u}_{i+1}(\boldsymbol{u}_i, \boldsymbol{u}_{i-1}) = (\tilde{\boldsymbol{d}}_i \times \boldsymbol{e}_{i+1,2}) \cdot (\boldsymbol{x}_i - \boldsymbol{p}_{i+1,0}),$$

$$\tilde{v}_{i+1}(\boldsymbol{u}_i, \boldsymbol{u}_{i-1}) = ((\boldsymbol{x}_i - \boldsymbol{p}_{i+1,0}) \times \boldsymbol{e}_{i+1,1}) \cdot \tilde{\boldsymbol{d}}_i,$$

$$\kappa_{i+1}(\boldsymbol{u}_i, \boldsymbol{u}_{i-1}) = (\tilde{\boldsymbol{d}}_i \times \boldsymbol{e}_{i+1,2}) \cdot \boldsymbol{e}_{i+1,1}.$$

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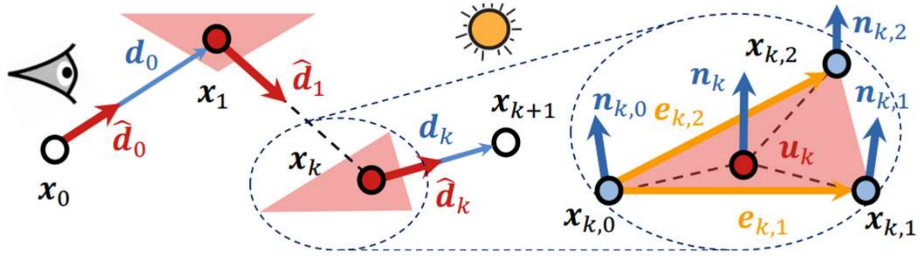


Möller–Trumbore

Reflection

- Accurate

$$\hat{\boldsymbol{d}}_i = -2(\hat{\boldsymbol{d}}_{i-1} \cdot \hat{\boldsymbol{n}}_i)\hat{\boldsymbol{n}}_i + \hat{\boldsymbol{d}}_{i-1}$$

through multiplying $\hat{\boldsymbol{d}}_i$ by $\boldsymbol{n}_i^2 \sqrt{\boldsymbol{d}_{i-1}^2}$:

$$\tilde{\boldsymbol{d}}_i = -2(\boldsymbol{d}_{i-1} \cdot \boldsymbol{n}_i)\boldsymbol{n}_i + \boldsymbol{d}_{i-1}\boldsymbol{n}_i^2$$

Refraction

$$\hat{\boldsymbol{d}}_i = \eta_i'(\hat{\boldsymbol{d}}_{i-1} - (\hat{\boldsymbol{d}}_{i-1} \cdot \hat{\boldsymbol{n}}_i)\hat{\boldsymbol{n}}_i) - \sqrt{1 - \eta_i'^2(1 - (\hat{\boldsymbol{d}}_{i-1} \cdot \hat{\boldsymbol{n}}_i)^2)}\hat{\boldsymbol{n}}_i$$

$$\tilde{\boldsymbol{d}}_i = \eta_i'(\boldsymbol{d}_{i-1}\boldsymbol{n}_i^2 - (\boldsymbol{d}_{i-1} \cdot \boldsymbol{n}_i)\boldsymbol{n}_i) - \sqrt{\beta_i}\boldsymbol{n}_i,$$

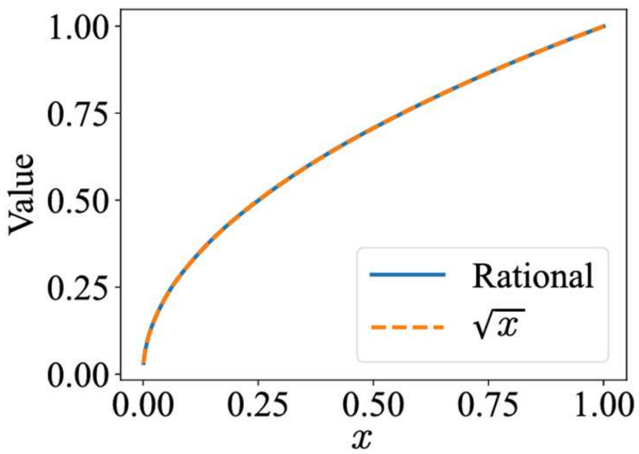
$$\beta_i = \boldsymbol{n}_i^2\boldsymbol{d}_{i-1}^2 - \eta_i'^2 \left(\boldsymbol{n}_i^2\boldsymbol{d}_{i-1}^2 - (\boldsymbol{d}_{i-1} \cdot \boldsymbol{n}_i)^2\right)$$

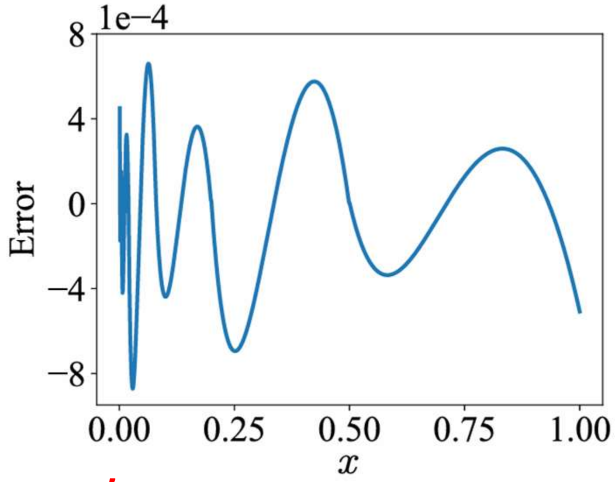
- Approximating $\sqrt{x}$ in $[0,1]$ with precision 10^{-3}

Refraction (cont'd)

- Approximating $\sqrt{x}$ in [0,1]

$$\frac{c_{0,i} + c_{1,i}x}{d_{0,i} + d_{1,i}x}, \ i = 0, 1, 2..., 5.$$





Why not polynomial? $(\sqrt{x})'|_{x=0} \to \infty$

Index	Left endpoint	Right endpoint	$c_{0,i}$	$c_{1,i}$	$d_{0,i}$	$d_{1,i}$	Max error
0	0.000	0.005	1.06939×10^{-1}	1.24883×10^{2}	6.44864	7.79412×10^{2}	0.0005
1	0.005	0.020	1.05021×10^{-1}	3.12337×10^{1}	3.20289	9.78683×10^{1}	0.0005
2	0.020	0.080	1.30984×10^{-1}	9.75997	2.00015	1.52961×10^{1}	0.0009
3	0.080	0.200	3.76068×10^{-1}	8.89489	3.19627	8.11322	0.0005
4	0.200	0.500	4.56906×10^{-1}	4.32322	2.45619	2.49402	0.0007
5	0.500	1.000	9.38873×10^{-1}	4.10143	3.41291	1.62996	0.0006

Milestone 2: Bivariate Specular Polynomials

	Type	Equation	#Var.	Degree
Multivar.	R^k	Eqs. (6), (13)	$2k$	2, 4
	T^k	Eqs. (6), (10)	$2k$	2, 6
	$(R\|T)^k$	Eq. (3)	$3k+1$	2
Bivar.	R	Eqs. (6), (13)	2	2, 4
	T	Eqs. (6), (10)	2	2, 6
	RR	Eqs. (6), (13), (14)	2	10, 16
	RT, TR	Eqs. (6), (10), (14)	2	10, 24
	TT	Eqs. (6), (13), (14)	2	18, 48

BVP → UVP

2 variables 1 variable

Resultant

y is hidden: $a_0 = a_0(y), a_1 = a_1(y)$

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$$

$$g(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_m$$

- The necessary condition of
  - the existence of common roots of
  - **two** (or more) polynomials

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & \cdots & a_{n-1} & a_n & \cdots & 0 \\ 0 & 0 & a_0 & \cdots & \cdots & a_{n-2} & a_{n-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_0 & \cdots & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & b_m & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & \cdots & b_{m-1} & b_m & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & \cdots & \cdots & b_m \end{vmatrix}$$

- Sylvester resultant
- Order: $n + m$

Bézout resultant

- Order: n

$$\begin{cases} a(u_1, v_1) = \sum_{i=0}^{n} a_i(v_1)u_1^i = 0, \\ b(u_1, v_1) = \sum_{i=0}^{n} b_i(v_1)u_1^i = 0. \end{cases}$$

$$r(v_1) = \det \boldsymbol{R}(v_1)$$

$$\sum_{k=0}^{\min(i,n-1-j)} \Big(a_{i-k}(v_1)b_{j+1+k}(v_1) - b_{i-k}(v_1)a_{j+1+k}(v_1)\Big)$$

A running example

$$\begin{cases} a(u_1, v_1) = u_1^3 + v_1^3 + u_1 v_1 - 1 = 0, \\ b(u_1, v_1) = u_1^2 + v_1^2 - 2 = 0. \end{cases} \tag{1}$$

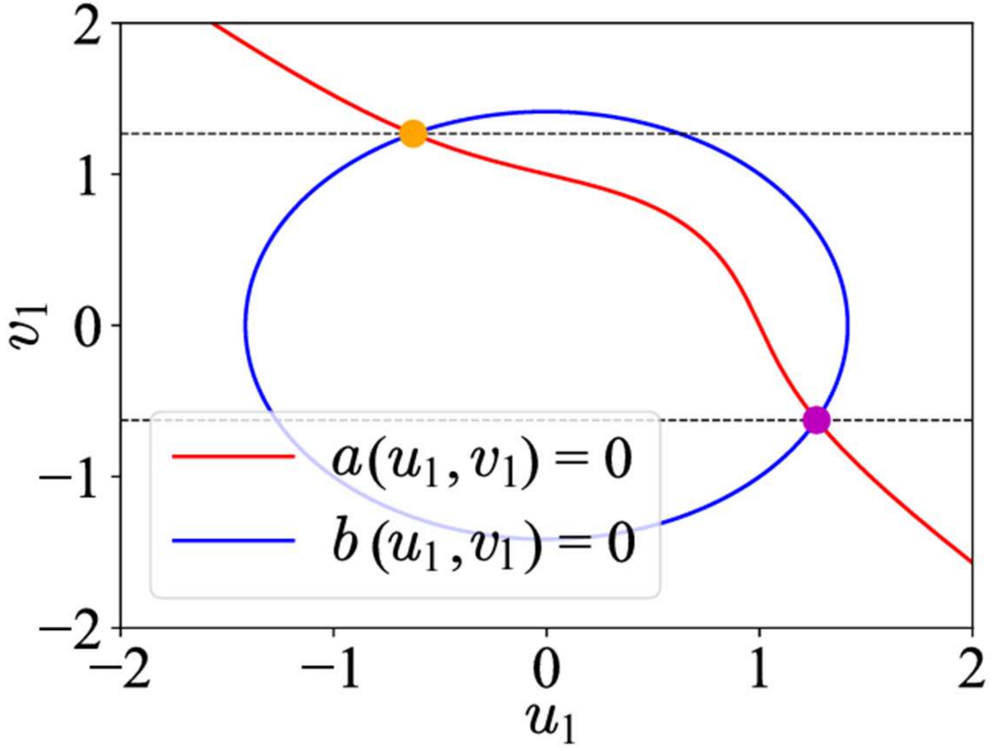
Its resultant matrix is

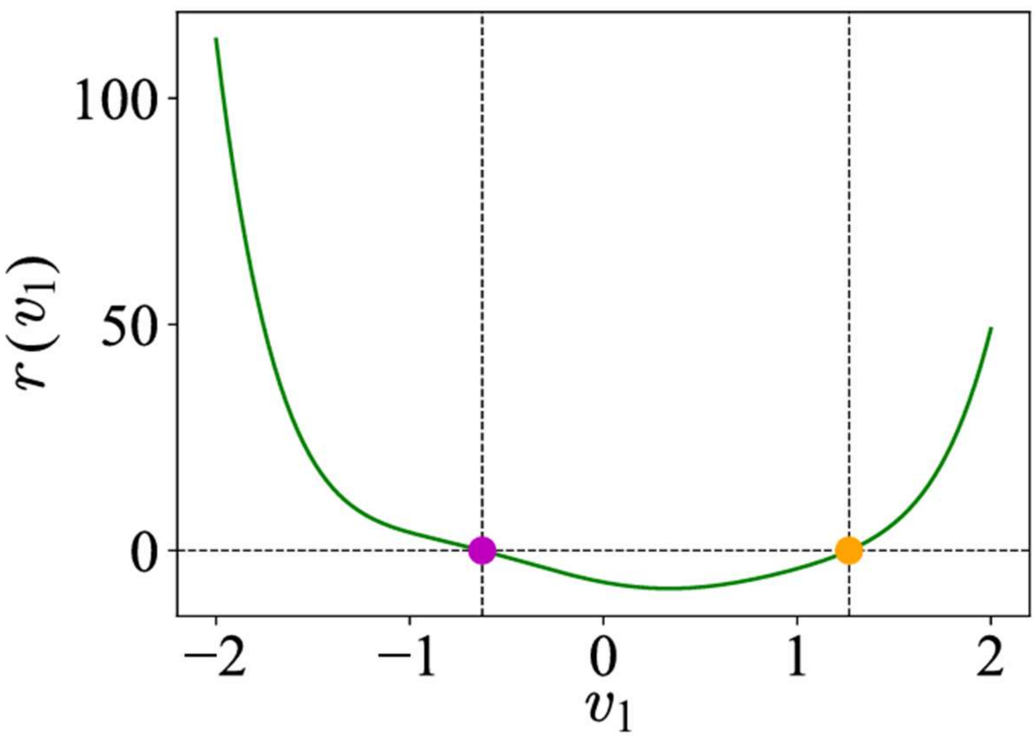
$$\boldsymbol{R}(v_1) = \begin{bmatrix} -v_1^3 + 2v_1 & v_1^3 - 1 & 2 - v_1^2 \\ v_1^3 - 1 & -v_1^2 + v_1 + 2 & 0 \\ 2 - v_1^2 & 0 & -1 \end{bmatrix}, \tag{2}$$

and the corresponding resultant is

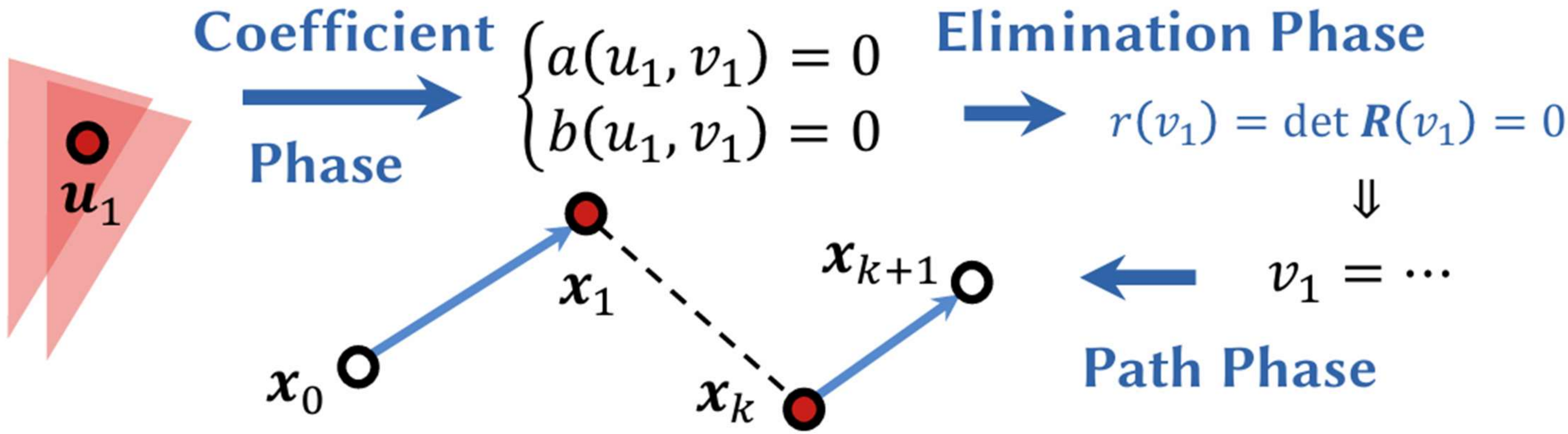
$$r(v_1) = \det \boldsymbol{R}(v_1) = 2v_1^6 - 2v_1^5 - 5v_1^4 + 6v_1^3 + 10v_1^2 - 8v_1 - 7. \tag{3}$$

A running example (cont'd)





Milestone 3: Univariate Specular Polynomials



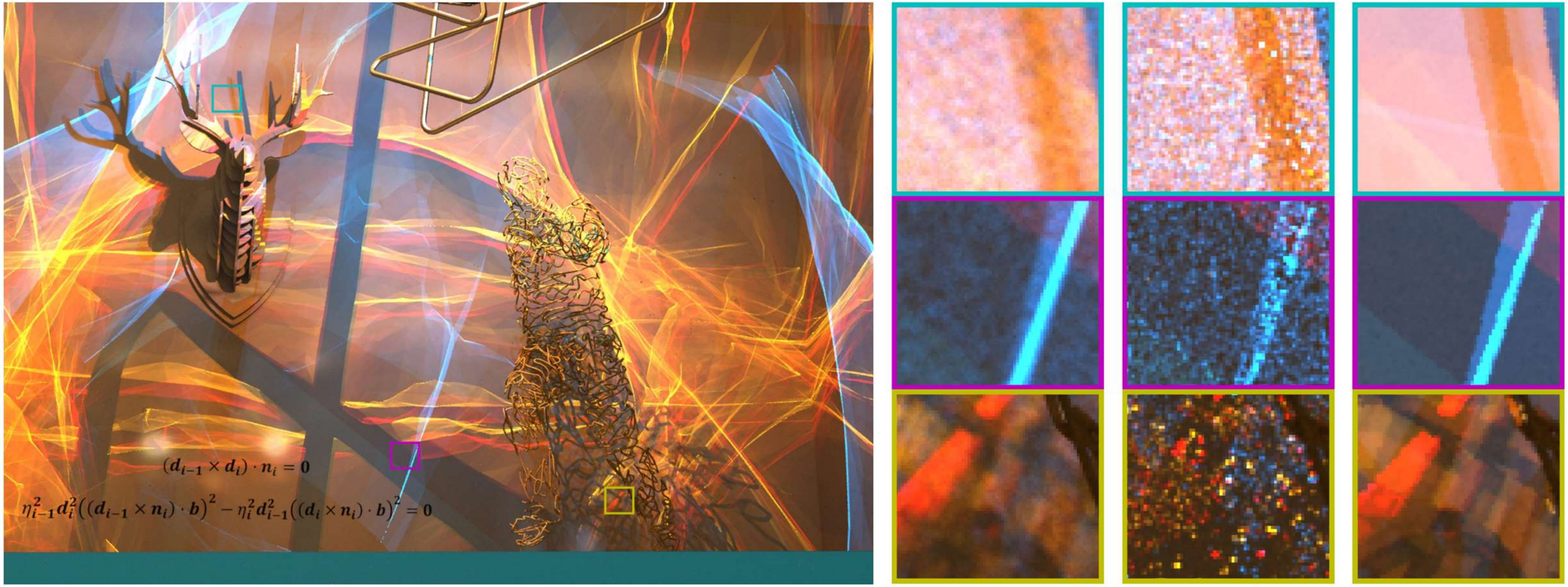
UVP → Roots

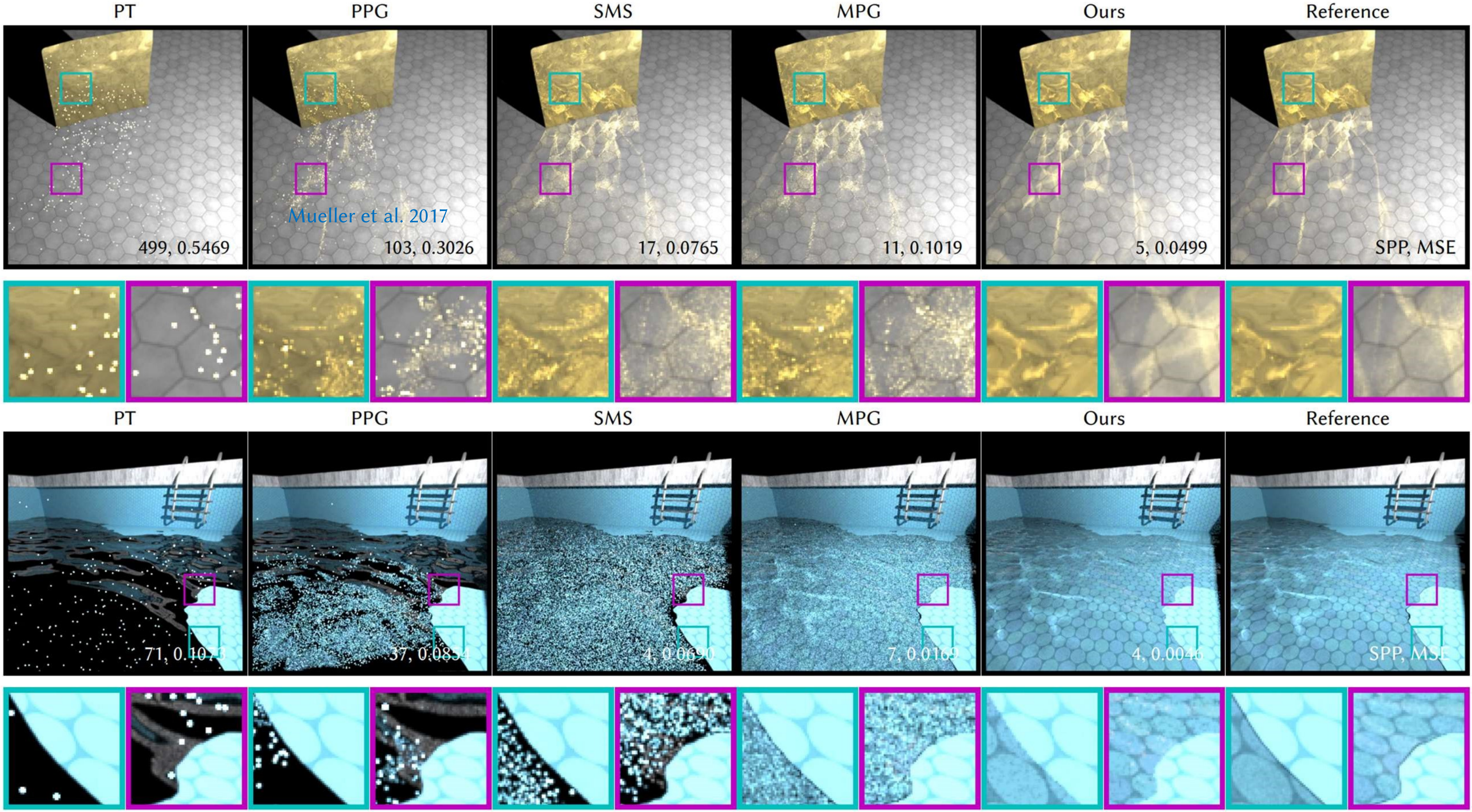
Please refer to the paper

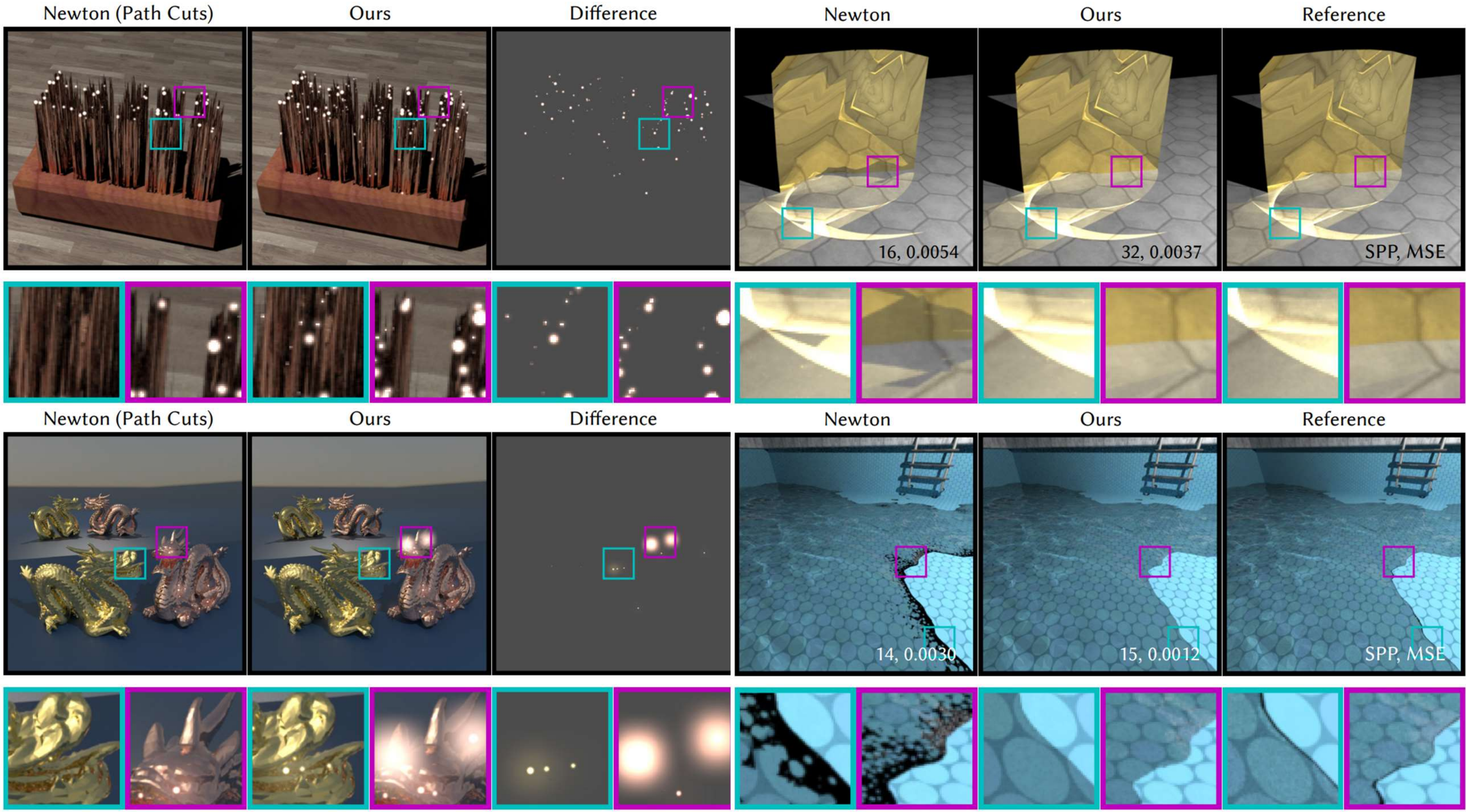
Equal-time comparison

Hachisuka et al. 2009 | Fan et al. 2023

SPPM | MPG | Ours




PT
PPG
SMS
MPG
Ours
Reference
Mueller et al. 2017
499, 0.5469
103, 0.3026
17, 0.0765
11, 0.1019
5, 0.0499
SPP, MSE
PT
PPG
SMS
MPG
Ours
Reference
71, 0.1073
37, 0.0854
4, 0.0690
7, 0.0169
4, 0.0046
SPP, MSE


Newton (Path Cuts)
Ours
Difference
Newton
Ours
Reference
16, 0.0054
32, 0.0037
SPP, MSE
Newton (Path Cuts)
Ours
Difference
Newton
Ours
Reference
14, 0.0030
15, 0.0012
SPP, MSE

Conclusion

$2k$ variable $\rightarrow$ 1 variable

- Reformulate the problem into univariate polynomial root-finding
- Good performance for single scattering

Future works (mostly for 2+ bounces)

- Reducing the degree of polynomials
- Numerical accuracy of solvers
- Reducing superfluous solutions
- Surface representations
- Accurate multiple refractions
- Glossy materials

Why polynomials?

Global convergence

Elimination to 1D

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THANK YOU